Solutions to Odd Exercises in *Game Physics, 2nd Edition*

David H. Eberly

April 13, 2010

Contents

A comment is in order about the examples and exercises in the book. Some of the reviewers of prepublication drafts for the first edition were critical of my inclusion of problems that you normally see in a standard physics textbook. Why cover them in a book entitled *Game Physics*? I have two reasons.

First, the misconception about the term "game physics" is that it refers entirely to hacks that are used in games in order to convince the player that the environment is behaving in a physically realistic manner. Moreover, many of the hacks are used to minimize the amount of CPU time spent on the physics. There is nothing wrong with hacks or minimizing CPU usage, but the world of computational physics involves more than just making it look right. The term "physical simulation" refers to building a virtual environment with simulated objects and simulated forces and torques. The objects move according to the laws of physics as determined by the equations of motion and subject to whatever physical constraints are present in the system. Commercial game physics engines provide you with the ability to obtain *physically correct behavior* or to use hacks to obtain *visually correct, physical behavior*. The former behavior requires you to understand physics and all its supporting topics. An understanding of problems presented in a standard physics textbook is sufficient to help you build a physical simulation, whether in a game application or any other type of application. The latter behavior itself will require at least a minimal knowledge of physical principles. I am presenting some standard problems because I believe they adequately convey the physical concepts that you will need in a game application as well as in other types of applications.

Second, as you will find out in the solutions to the exercises, many physics textbooks stop short of showing you how to actually *compute* object motion. The textbooks tend to develop mathematical equations that are not the ones you need to use in a computer implementation. My exercises are designed to bridge the gap between deriving standard physics textbook equations and implementing them on a computer. Sometimes this gap is larger than you might expect. In my opinion, a classical problem that illustrates how large the gap can be is Exercise 3.3 on Kepler's Laws. If you try solving equation (3.1) directly with a numerical differential equation solver, you will likely have severe problems with stability of the solution. Equation (3.2) is the end of the road for a textbook presentation. The conclusion is that the Earth travels about the Sun in an elliptical path. The equation tells you how the distance r from the Sun is related to the polar angle θ, and the equation has various parameters of interest, but it gives you no clue about how the polar angle varies with time or how to compute the parameters. My solution to that exercise shows you the additional mathematical steps to reformulate the problem so that you can solve it in a numerically stable manner. The CDROM has an implementation, the last step in the process of building a real physical simulation.

1 Introducton

No exercises.

2 Basic Concepts from Physics

Exercise 2.1. The position is

$$\mathbf{r}(t) = \begin{cases} (t, t^3, 0), & t < 0 \\ (t, 0, t^3), & t \geq 0 \end{cases}$$

the velocity is

$$\mathbf{v}(t) = \begin{cases} (1, 3t^2, 0), & t < 0 \\ (1, 0, 3t^2), & t \geq 0 \end{cases}$$

and the acceleration is

$$\mathbf{a}(t) = \begin{cases} (0, 6t, 0), & t < 0 \\ (0, 0, 6t), & t \geq 0 \end{cases}$$

The limit as t approaches zero from values smaller than zero is

$$\lim_{t \to 0^-} \mathbf{r}(t) = \lim_{t \to 0^-} (t, t^3, 0) = \left(\lim_{t \to 0^-} t, \lim_{t \to 0^-} t^3, \lim_{t \to 0^-} 0 \right) = (0, 0, 0)$$

The last equality is true since the component functions are polynomials in t. Polynomials are continuous functions, so limits are computed just by evaluating the polynomials at the value that t approaches. The limit as t approaches zero from values larger than zero is

$$\lim_{t \to 0^+} \mathbf{r}(t) = \lim_{t \to 0^+} (t, 0, t^3) = \left(\lim_{t \to 0^+} t, \lim_{t \to 0^+} 0, \lim_{t \to 0^+} t^3 \right) = (0, 0, 0)$$

Again, the last equality is true since polynomial functions are continuous. A similar argument applies to velocity and acceleration. The limiting vector for velocity is $(1, 0, 0)$ and the limiting vector for acceleration is $(0, 0, 0)$.

Notice that $|\mathbf{v}(t)| = |(1, 3t^2, 0)| = \sqrt{1 + 9t^4}$ for $t < 0$ and $|\mathbf{v}(t)| = |(1, 0, 3t^2)| = \sqrt{1 + 9t^4}$ for $t \geq 0$. The speed function is

$$\dot{s}(t) = \sqrt{1 + 9t^4}$$

for all t. Consider the portion of the curve for $t < 0$. This portion is a curve in the xy-plane with tangent vector

$$\mathbf{T}(t) = \frac{(1, 3t^2, 0)}{\sqrt{1 + 9t^4}}$$

A unit-length normal according to equation (2.5) is

$$\mathbf{N}(t) = \frac{(-3t^2, 1, 0)}{\sqrt{1 + 9t^4}}$$

and has limit

$$\lim_{t \to 0^-} \mathbf{N}(t) = \lim_{t \to 0^-} \frac{(-3t^2, 1, 0)}{\sqrt{1 + 9t^4}} = (0, 1, 0)$$

The components of the normal vector are all continuous functions of t, so the limit is computed by simply evaluating the components.

Now consider the portion of the curve for $t \geq 0$. This portion is a curve in the xz-plane with tangent vector

$$\mathbf{T}(t) = \frac{(1, 0, 3t^2)}{\sqrt{1 + 9t^4}}$$

A unit-length normal according to equation (2.5) is

$$\mathbf{N}(t) = \frac{(-3t^2, 0, 1)}{\sqrt{1 + 9t^4}}$$

and has limit

$$\lim_{t \to 0^+} \mathbf{N}(t) = \lim_{t \to 0^+} \frac{(-3t^2, 0, 1)}{\sqrt{1 + 9t^4}} = (0, 0, 1)$$

The one-sided limits of $\mathbf{N}(t)$ at $t = 0$ are different. Even a sign change on one of the normals cannot force the one-sided limits to be equal, so it is not possible to define a continuous function for the curve normal vector at $t = 0$.

Exercise 2.3. The spherical helix is

$$(x, y, z) = \frac{(\cos t, \sin t, t)}{\sqrt{1+t^2}}$$

Notice that $\rho = |(x, y, z)| = 1$ for all t, so this curve does in fact lie on a sphere of radius 1. Using the equations for spherical coordinates,

$$\frac{\cos t}{\sqrt{1+t^2}} = x = \cos\theta \sin\phi, \quad \frac{\sin t}{\sqrt{1+t^2}} = y = \sin\theta \sin\phi, \quad \frac{t}{\sqrt{1+t^2}} = z = \cos\phi$$

The last equation implies $\sin\phi = 1/\sqrt{1+t^2}$. Replacing this in the first two equations and cancelling common terms leads to $\cos\theta = \cos t$ and $\sin\theta = \sin t$, so we may choose $\theta(t) = t$. In summary, we have

$$\rho(t) = 1, \;\; \theta(t) = t, \;\; \phi(t) = \cos^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$$

where the branch of the inverse cosine has values in $[0, \pi]$. The relevant derivatives are $\dot{\rho} = 0$, $\ddot{\rho} = 0$, $\dot{\theta} = 1$, and $\ddot{\theta} = 0$. The derivatives of ϕ use the identities

$$\frac{d}{du}\cos^{-1}(u) = \frac{-1}{\sqrt{1-u^2}}, \quad \frac{d}{dt}\frac{t}{\sqrt{1+t^2}} = \frac{1}{(1+t^2)^{3/2}}$$

and the chain rule:

$$\dot{\phi} = \frac{d}{dt}\cos^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right) = \frac{-1}{\sqrt{1-\frac{t^2}{1+t^2}}}\,\frac{1}{(1+t^2)^{3/2}} = \frac{-1}{1+t^2}$$

and

$$\ddot{\phi} = \frac{2t}{(1+t^2)^2}$$

Using equations (2.30), (2.31), and (2.32) for spherical coordinates, the position is

$$\mathbf{r} = \mathbf{R}$$

the velocity is

$$\mathbf{v} = (\sin\phi)\mathbf{P} - (\dot{\phi})\mathbf{Q} = \left(\frac{1}{\sqrt{1+t^2}}\right)\mathbf{P} - \left(\frac{1}{1+t^2}\right)\mathbf{Q}$$

and the acceleration is

$$\mathbf{a} = (2\dot{\phi}\cos\phi)\mathbf{P} + (\sin\phi\cos\phi - \ddot{\phi})\mathbf{Q} - (\dot{\phi}^2 + \sin^2\phi)\mathbf{R} = -\frac{2t}{(1+t^2)^{3/2}}\mathbf{P} - \frac{t}{1+t^2}\mathbf{Q} - \frac{2+t^2}{(1+t^2)^2}\mathbf{R}$$

The (x, y) components of the curve satisfy $r^2 = x^2 + y^2 = 1/(1+t^2)$. As t becomes infinite, r goes to zero. However, x has a cosine term and y has a sine term, so the path in the xy-plane is a spiral into the origin. The z component has a limit,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty}\frac{t}{\sqrt{1+t^2}} = \sqrt{\lim_{t\to\infty}\frac{t^2}{1+t^2}} = \sqrt{\lim_{t\to\infty}\left(1 - \frac{1}{1+t^2}\right)} = \sqrt{1} = 1$$

On the sphere itself, the spherical helix spirals about the north pole $(0, 0, 1)$ and reaches it in the limit as t becomes infinite.

Exercise 2.5. The relationship between the angular velocity $\mathbf{w}(t)$ and the rotation axis $\mathbf{D}$ is

$$\text{Skew}(\mathbf{w}) = \dot{R}R^{\mathrm{T}}$$

where

$$R = I + \sigma S + (1-\gamma)S^2, \;\; R^{\mathrm{T}} = I - \sigma S + (1-\gamma)S^2$$

with $S = \text{Skew}(\mathbf{D})$, $\sigma = \sin(\theta)$, and $\gamma = \cos(\theta)$. Observe that $\dot{S} = \text{Skew}(\dot{\mathbf{D}})$. The derivative of the rotation matrix is

$$\dot{R} = \sigma\dot{S} + \dot{\theta}\gamma S + (1-\gamma)(S\dot{S} + \dot{S}S) + \dot{\theta}\sigma S^2$$

Matrix multiplication is not commutative in general, so beware not to use the power rule for derivatives. That is, it is not generally the case that $d(S^2)/dt = 2S\dot{S}$.

An invertible matrix M is uniquely determine by how it acts on three linearly independent vectors. Moreover, it is convenient to use three unit-length and mutually perpendicular vectors. Specifically, let $\mathbf{U}_1$, $\mathbf{U}_2$, and $\mathbf{U}_3$ be unit length and mutually perpendicular with $\mathbf{U}_3 = \mathbf{U}_1 \times \mathbf{U}_2$. Let $\mathbf{A}_i = M\mathbf{U}_i$ for $1 \le i \le 3$. Write these as a single matrix equation,

$$MU = M[\mathbf{U}_1 \; \mathbf{U}_2 \; \mathbf{U}_3] = [M\mathbf{U}_1 \; M\mathbf{U}_2 \; M\mathbf{U}_3] = [\mathbf{A}_1 \; \mathbf{A}_2 \; \mathbf{A}_3] = A$$

where U and A are matrices whose columns are those shown. By our assumptions on the $\mathbf{U}_i$, U is a rotation matrix. Its inverse is just its transpose, so

$$M = AU^{\mathrm{T}}$$

For our problem at hand, we want to compute $\dot{R}R^{\mathrm{T}}$. The unit-length and mutually perpendicular vectors we will use are

$$\mathbf{U}_1 = \mathbf{D}, \;\; \mathbf{U}_2 = \dot{\mathbf{D}}, \;\; \mathbf{U}_3 = \mathbf{D} \times \dot{\mathbf{D}}$$

In the constructions we use the following identities

$$\begin{aligned}
&\mathbf{U}_1 = \mathbf{U}_2 \times \mathbf{U}_3, \;\; \mathbf{U}_2 = \mathbf{U}_3 \times \mathbf{U}_1, \;\; \mathbf{U}_3 = \mathbf{U}_1 \times \mathbf{U}_2, \\
&S\mathbf{U}_1 = \mathbf{0}, \;\; S\mathbf{U}_2 = \mathbf{U}_3. \;\; S\mathbf{U}_3 = -\mathbf{U}_2, \\
&\dot{S}\mathbf{U}_1 = -\mathbf{U}_3, \;\; \dot{S}\mathbf{U}_2 = \mathbf{0}, \;\; \dot{S}\mathbf{U}_3 = \mathbf{U}_1
\end{aligned}$$

First, let us apply just the R^{T} portion to the vectors.

$$\begin{aligned}
R^{\mathrm{T}}\mathbf{U}_1 &= \mathbf{U}_1 - \sigma S\mathbf{U}_1 + (1-\gamma)S^2\mathbf{U}_1 = \mathbf{U}_1 \\
R^{\mathrm{T}}\mathbf{U}_2 &= \mathbf{U}_2 - \sigma S\mathbf{U}_2 + (1-\gamma)S^2\mathbf{U}_2 = \mathbf{U}_2 - \sigma\mathbf{U}_3 - (1-\gamma)\mathbf{U}_2 = \gamma\mathbf{U}_2 - \sigma\mathbf{U}_3 \\
R^{\mathrm{T}}\mathbf{U}_3 &= \mathbf{U}_3 - \sigma S\mathbf{U}_3 + (1-\gamma)S^2\mathbf{U}_3 = \mathbf{U}_3 + \sigma\mathbf{U}_2 - (1-\gamma)\mathbf{U}_3 = \sigma\mathbf{U}_2 + \gamma\mathbf{U}_3
\end{aligned}$$

Second, apply $\dot{R}$ to the original vectors.

$$\begin{aligned}
\dot{R}\mathbf{U}_1 &= (\sigma\dot{S} + \dot{\theta}\gamma S + (1-\gamma)(S\dot{S} + \dot{S}S) + \dot{\theta}\sigma S^2)\mathbf{U}_1 = (1-\gamma)\mathbf{U}_2 - \sigma\mathbf{U}_3 \\
\dot{R}\mathbf{U}_2 &= (\sigma\dot{S} + \dot{\theta}\gamma S + (1-\gamma)(S\dot{S} + \dot{S}S) + \dot{\theta}\sigma S^2)\mathbf{U}_2 = (1-\gamma)\mathbf{U}_1 - \dot{\theta}\sigma\mathbf{U}_2 + \dot{\theta}\gamma\mathbf{U}_3 \\
\dot{R}\mathbf{U}_3 &= (\sigma\dot{S} + \dot{\theta}\gamma S + (1-\gamma)(S\dot{S} + \dot{S}S) + \dot{\theta}\sigma S^2)\mathbf{U}_3 = \sigma\mathbf{U}_1 - \dot{\theta}\gamma\mathbf{U}_2 - \dot{\theta}\sigma\mathbf{U}_3
\end{aligned}$$

Third, compose the two operations,

$$\dot{R}R^{\mathrm{T}}\mathbf{U}_1 = \dot{R}\mathbf{U}_1 = (1-\gamma)\mathbf{U}_2 - \sigma\mathbf{U}_3$$
$$\dot{R}R^{\mathrm{T}}\mathbf{U}_2 = \dot{R}(\gamma\mathbf{U}_2 - \sigma\mathbf{U}_3) = -(1-\gamma)\mathbf{U}_1 + \dot{\theta}\mathbf{U}_3$$
$$\dot{R}R^{\mathrm{T}}\mathbf{U}_3 = \dot{R}(\sigma\mathbf{U}_2 + \gamma\mathbf{U}_3) = \sigma\mathbf{U}_1 - \dot{\theta}\mathbf{U}_2$$

Setting $U = [\mathbf{U}_1\ \mathbf{U}_2\ \mathbf{U}_3]$ and factoring leads to

$$\dot{R}R^{\mathrm{T}}U = [(1-\gamma)\mathbf{U}_2 - \sigma\mathbf{U}_3 \mid -(1-\gamma)\mathbf{U}_1 + \dot{\theta}\mathbf{U}_3 \mid \sigma\mathbf{U}_1 - \dot{\theta}\mathbf{U}_2]$$

Inverting U to the right-hand side,

$$\begin{aligned}
\dot{R}R^{\mathrm{T}} &= [(1-\gamma)\mathbf{U}_2 - \sigma\mathbf{U}_3 \mid -(1-\gamma)\mathbf{U}_1 + \dot{\theta}\mathbf{U}_3 \mid \sigma\mathbf{U}_1 - \dot{\theta}\mathbf{U}_2]\begin{bmatrix}\mathbf{U}_1^{\mathrm{T}}\\ \mathbf{U}_2^{\mathrm{T}}\\ \mathbf{U}_3^{\mathrm{T}}\end{bmatrix}\\
&= \dot{\theta}\left(\mathbf{U}_3\mathbf{U}_2^{\mathrm{T}} - \mathbf{U}_2\mathbf{U}_3^{\mathrm{T}}\right) + \sigma\left(\mathbf{U}_1\mathbf{U}_3^{\mathrm{T}} - \mathbf{U}_3\mathbf{U}_1^{\mathrm{T}}\right) + (1-\gamma)\left(\mathbf{U}_2\mathbf{U}_1^{\mathrm{T}} - \mathbf{U}_1\mathbf{U}_2^{\mathrm{T}}\right)
\end{aligned}$$

The last step involves verifying a relationship between the matrices in the last equation and the skew operator. Let $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$. Then

$$\begin{aligned}
\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}} &= \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}\begin{bmatrix}a_1 & a_2 & a_3\end{bmatrix} - \begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}\begin{bmatrix}b_1 & b_2 & b_3\end{bmatrix}\\
&= \begin{bmatrix}b_1a_1 & b_1a_2 & b_1a_3\\ b_2a_1 & b_2a_2 & b_2a_3\\ b_3a_1 & b_3a_2 & b_3a_3\end{bmatrix} - \begin{bmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ a_2b_1 & a_2b_2 & a_2b_3\\ a_3b_1 & a_3b_2 & a_3b_3\end{bmatrix}\\
&= \begin{bmatrix}0 & -(a_1b_2 - a_2b_1) & +(a_3b_1 - a_1b_3)\\ +(a_1b_2 - a_2b_1) & 0 & -(a_2b_3 - a_3b_2)\\ -(a_3b_1 - a_1b_3) & +(a_2b_3 - a_3b_2) & 0\end{bmatrix}\\
&= \mathrm{Skew}(\mathbf{A}\times\mathbf{B})
\end{aligned}$$

Applying this identity to our equation for $\dot{R}R^{\mathrm{T}}$, we have

$$\begin{aligned}
\mathrm{Skew}(\mathbf{w}) &= \dot{R}R^{\mathrm{T}}\\
&= \dot{\theta}\,\mathrm{Skew}(\mathbf{U}_2\times\mathbf{U}_3) + \sigma\,\mathrm{Skew}(\mathbf{U}_3\times\mathbf{U}_1) + (1-\gamma)\,\mathrm{Skew}(\mathbf{U}_1\times\mathbf{U}_2)\\
&= \dot{\theta}\,\mathrm{Skew}(\mathbf{U}_1) + \sigma\,\mathrm{Skew}(\mathbf{U}_2) + (1-\gamma)\,\mathrm{Skew}(\mathbf{U}_3)\\
&= \dot{\theta}\,\mathrm{Skew}\mathbf{D} + (\sin\theta)\,\mathrm{Skew}(\dot{\mathbf{D}}) + (1-\cos\theta)\,\mathrm{Skew}(\mathbf{D}\times\dot{\mathbf{D}})
\end{aligned}$$

Using the fact that $\mathrm{Skew}(\mathbf{A}) = \mathrm{Skew}(\mathbf{B})$ implies $\mathbf{A} = \mathbf{B}$, we may remove the skew operator to obtain the desired result,

$$\mathbf{w} = \dot{\theta}\mathbf{D} + (\sin\theta)\dot{\mathbf{D}} + (1 - \cos\theta)\mathbf{D} \times \dot{\mathbf{D}}$$

Exercise 2.7. The position is

$$\mathbf{r} = \frac{((1-t^2)\cos(\pi t), (1-t^2)\sin(\pi t), t^2)}{\sqrt{(1-t^2)^2 + t^4}}$$

We need to define an orientation matrix for the rigid sphere. We know that $|\mathbf{r}| = 1$ for all t, so the position vector itself may be used as one of the columns. Moreover, the condition $\mathbf{r}\cdot\mathbf{r} = 1$ for all t implies $\mathbf{r}\cdot\dot{\mathbf{r}} = 0$ for all t, in which case $\dot{\mathbf{r}}$ is always perpendicular to $\mathbf{r}$. Another column is $\mathbf{u} = \dot{\mathbf{r}}/|\dot{\mathbf{r}}|$. The final column is the cross product of the other two columns. The orientation matrix is

$$R(t) = \left[\begin{array}{c|c|c} \mathbf{r} & \mathbf{u} & \mathbf{r}\times\mathbf{u} \end{array}\right]$$

The angular velocity vector is define by equation (2.38), but is solved explicitly as

$$\mathrm{Skew}(\mathbf{w}) = \dot{R}R^{\mathrm{T}}$$

We need only compute the right-hand side using the orientation matrix we have chosen.

The quotient rule may be applied to $\mathbf{u}$ to obtain its derivative,

$$\dot{\mathbf{u}} = \frac{|\dot{\mathbf{r}}|^2\ddot{\mathbf{r}} - (\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^3} = \frac{\ddot{\mathbf{r}} - (\ddot{\mathbf{r}}\cdot\mathbf{u})\mathbf{u}}{|\dot{\mathbf{r}}|}$$

The product rule may be applied to $\mathbf{r}\times\mathbf{u}$ to obtain its derivative

$$\frac{d(\mathbf{r}\times\mathbf{u})}{dt} = \mathbf{r}\times\dot{\mathbf{u}} + \dot{\mathbf{r}}\times\mathbf{u} = \mathbf{r}\times\dot{\mathbf{u}}$$

The last equality is true since the velocity and the normalized velocity vectors are parallel, in which case the cross product of the two is the zero vector. The derivative of the orientation matrix is

$$\dot{R} = \left[\begin{array}{c|c|c} \dot{\mathbf{r}} & \dot{\mathbf{u}} & \mathbf{r}\times\dot{\mathbf{u}} \end{array}\right]$$

The angular velocity is

$$\mathrm{Skew}(\mathbf{w}) = \dot{R}R^{\mathrm{T}} = \left[\begin{array}{c|c|c} \dot{\mathbf{r}} & \dot{\mathbf{u}} & \mathbf{r}\times\dot{\mathbf{u}} \end{array}\right] \left[\begin{array}{c} \mathbf{r} \\ \hline \mathbf{u} \\ \hline \mathbf{r}\times\mathbf{u} \end{array}\right] = \dot{\mathbf{r}}\mathbf{r}^{\mathrm{T}} + \dot{\mathbf{u}}\mathbf{u}^{\mathrm{T}} + (\mathbf{r}\times\dot{\mathbf{u}})(\mathbf{r}\times\mathbf{u})^{\mathrm{T}}$$

Just as we defined the skew-symmetric matrix $S = \mathrm{Skew}(\mathbf{w})$, we may define the inverse operations $\mathbf{w} = \mathrm{Unskew}(S)$. Therefore,

$$\mathbf{w} = \mathrm{Unskew}\left(\dot{\mathbf{r}}\mathbf{r}^{\mathrm{T}} + \dot{\mathbf{u}}\mathbf{u}^{\mathrm{T}} + (\mathbf{r}\times\dot{\mathbf{u}})(\mathbf{r}\times\mathbf{u})^{\mathrm{T}}\right)$$

Although you could go through the horrendous algebraic details of computing the vectors $\dot{\mathbf{r}}$, $\mathbf{u}$, and $\dot{\mathbf{u}}$ for the specific position $\mathbf{r}$ of this problem, all that is needed for a computer implementation are functions for $\mathbf{r}$, $Vectorv = \dot{\mathbf{r}}$, and $\mathbf{a} = \ddot{\mathbf{r}}$. These functions are somewhat complicated in the current example, so you could always rely on a symbolic mathematics program to generate the C code for you (Mathematica can do this).

```
Vector3d RFunction (double t) { /* position calculations */ }
Vector3d VFunction (double t) { /* velocity calculations */ }
Vector3d AFunction (double t) { /* acceleration calculations */ }

void ComputeTerms (double t, Vector3d& R, Vector3d& V, Vector3d& U, Vector3d& UDot)
{
    R = RFunction(t);
    V = VFunction(t);
    Vector3d A = AFunction(t);
    double length = V.Length();
    U = V/length;
    UDot = (A - R.Dot(U)*U)/length;
}

Vector3d WFunction (double t)
{
    Vector3d R, V, U, UDot;
    ComputeTerms(t,R,V,A,U,UDot);

    Vector3d RxU = R.Cross(U);
    Vector3d RxUDot = R.Cross(UDot);

    Vector3d W;
    W.X() = -(V.Y()*R.Z() + UDot.Y()*U.Z() + RxUDot.Y()*RxU.Z());
    W.Y() = +(V.X()*R.Z() + UDot.X()*U.Z() + RxUDot.X()*RxU.Z());
    W.Z() = -(V.X()*R.Y() + UDot.X()*U.Y() + RxUDot.X()*RxU.Y());
    return W;
}
```

Exercise 2.9. Suppose that the line of the fulcrum has unit-length direction vector $\mathbf{D}$ and unit-length normal vector $\mathbf{N}$. The line contains the center of mass $\mathbf{C} = (\bar{x}, \bar{y})$. In parameteric form, the line is $\mathbf{P}(s) = \mathbf{C} + s\mathbf{D}$ for any real number s. In normal form, the line is $\mathbf{N} \cdot (\mathbf{P} - \mathbf{C}) = 0$.

The center of mass, direction, and normal form a two-dimensional coordinate system. A particle located at $\mathbf{P}_i = (x_i, y_i)$ may be written as

$$\mathbf{P}_i = \mathbf{C} + s_i\mathbf{D} + t_i\mathbf{N}$$

where

$$s_i = \mathbf{D} \cdot (\mathbf{P}_i - \mathbf{C})$$

and

$$t_i = \mathbf{N} \cdot (\mathbf{P}_i - \mathbf{C})$$

Observe that t_i is the distance from $\mathbf{P}_i$ to the fulcrum line. The first moment of the system about the fulcrum line is

$$\begin{aligned} M &= \textstyle\sum_{i=1}^{p} m_i t_i \\ &= \textstyle\sum_{i=1}^{p} m_i \mathbf{N} \cdot (\mathbf{P}_i - \mathbf{C}) \\ &= \textstyle\sum_{i=1}^{p} m_i \mathbf{N} \cdot (x_i - \bar{x}, y_i - \bar{y}) \\ &= \mathbf{N} \cdot \textstyle\sum_{i=1}^{p} m_i (x_i - \bar{x}, y_i - \bar{y}) \\ &= \mathbf{N} \cdot (0, 0) \\ &= 0 \end{aligned}$$

The fact that $\sum_{i=1}^{p} m_i (x_i - \bar{x}, y_i - \bar{y}) = (0, 0)$ was already discussed in the book. Since the first moment is zero with respect to the fulcrum line, the plate must balance on the fulcrum. This is true regardless of the line orientation as specified by $\mathbf{N}$.

Exercise 2.11

It should be intuitive that the result is true. The rotation is a rigid motion. If you imagine $\{\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b}\}$ as a nonorthogonal coordinate frame, the rotation preserves all the relationships among these vectors.

Here is one way to prove the identity. Another approach is shown in Exercise 5.8.

Let $\mathbf{w}$ be a unit-length vector for the axis of rotation. Let the angle of rotation be θ. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors for which the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a right-handed orthonormal set. That is, the vectors are unit length, mutually perpendicular; also, $\mathbf{u} = \mathbf{v}\times\mathbf{w}$, $\mathbf{v} = \mathbf{w}\times\mathbf{u}$, and $\mathbf{w} = \mathbf{u}\times\mathbf{v}$. The rotation acts on the vectors according to the following:

$$R\mathbf{u} = c\mathbf{u} + s\mathbf{v}, \quad R\mathbf{v} = -s\mathbf{u} + c\mathbf{v}, \quad R\mathbf{w} = \mathbf{w}$$

where $c = \cos\theta$ and $s = \sin\theta$.

Represent the specified vectors in the coordinate system of this orthonormal set.

$$\begin{aligned}
\mathbf{a} &= \alpha_0\mathbf{u} + \alpha_1\mathbf{v} + \alpha_2\mathbf{w} \\
\mathbf{b} &= \beta_0\mathbf{u} + \beta_1\mathbf{v} + \beta_2\mathbf{w} \\
\mathbf{a}\times\mathbf{b} &= \gamma_0\mathbf{u} + \gamma_1\mathbf{v} + \gamma_2\mathbf{w}
\end{aligned}$$

where

$$(\gamma_0, \gamma_1, \gamma_2) = (\alpha_1\beta_2 - \alpha_2\beta_1, \alpha_2\beta_0 - \alpha_0\beta_2, \alpha_0\beta_1 - \alpha_1\beta_0)$$

Then

$$\begin{aligned}
R\mathbf{a} &= \alpha_0 R\mathbf{u} + \alpha_1 R\mathbf{v} + \alpha_2 R\mathbf{w} = (c\alpha_0 - s\alpha_1)\mathbf{u} + (s\alpha_0 + c\alpha_1)\mathbf{v} + \alpha_2\mathbf{w} \\
R\mathbf{b} &= \beta_0 R\mathbf{u} + \beta_1 R\mathbf{v} + \beta_2 R\mathbf{w} = (c\beta_0 - s\beta_1)\mathbf{u} + (s\beta_0 + c\beta_1)\mathbf{v} + \beta_2\mathbf{w} \\
R(\mathbf{a}\times\mathbf{b}) &= \gamma_0 R\mathbf{u} + \gamma_1 R\mathbf{v} + \gamma_2 R\mathbf{w} = (c\gamma_0 - s\gamma_1)\mathbf{u} + (s\gamma_0 + c\gamma_1)\mathbf{v} + \gamma_2\mathbf{w}
\end{aligned}$$

Computing in the coordinate system of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, some algebraic manipulations will show that

$$\begin{aligned}
(R\mathbf{a})\times(R\mathbf{b}) &= (c\alpha_0 - s\alpha_1, s\alpha_0 + c\alpha_1, \alpha_2)\times(c\beta_0 - s\beta_1, s\beta_0 + c\beta_1, \beta_2) \\
&= (c(\alpha_1\beta_2 - \alpha_2\beta_1) - s(\alpha_2\beta_0 - \alpha_0\beta_2), c(\alpha_2\beta_0 - \alpha_0\beta_2) + s(\alpha_1\beta_2 - \alpha_2\beta_1), \alpha_0\beta_1 - \alpha_1\beta_0)
\end{aligned}$$

It is also straightforward to show that

$$\begin{aligned}
R(\mathbf{a}\times\mathbf{b}) &= (c\gamma_0 - s\gamma_1, s\gamma_0 + c\gamma_1, \gamma_2) \\
&= (c(\alpha_1\beta_2 - \alpha_2\beta_1) - s(\alpha_2\beta_0 - \alpha_0\beta_2), c(\alpha_2\beta_0 - \alpha_0\beta_2) + s(\alpha_1\beta_2 - \alpha_2\beta_1), \alpha_0\beta_1 - \alpha_1\beta_0)
\end{aligned}$$

The two expressions are equal.

Exercise 2.13. The region in Example 2.3 is parameterized by x and y where $x^2 \leq y \leq 1$, $-1 \leq x \leq 1$, and $z = 0$. The mass density is assumed to be 1. The first inertia tensor component is

$$\begin{aligned} I_{xx} &= \int_{-1}^{1} \int_{x^2}^{1} y^2 + z^2 \, dy \, dx \\ &= \int_{-1}^{1} \int_{x^2}^{1} y^2 \, dy \, dx \\ &= \int_{-1}^{1} \frac{y^3}{3} \bigg|_{x^2}^{1} dx \\ &= \int_{-1}^{1} \frac{1-x^6}{3} \, dx \\ &= \frac{x - x^7/7}{3} \bigg|_{-1}^{1} \\ &= \frac{4}{7} \end{aligned}$$

The constructions are similar for the other components.

$$\begin{aligned} I_{yy} &= \int_{-1}^{1} \int_{x^2}^{1} x^2 + z^2 \, dy \, dx = \frac{4}{15} \\ I_{zz} &= \int_{-1}^{1} \int_{x^2}^{1} x^2 + y^2 \, dy \, dx = \frac{88}{105} \\ I_{xy} &= \int_{-1}^{1} \int_{x^2}^{1} xy \, dy \, dx = 0 \\ I_{xz} &= \int_{-1}^{1} \int_{x^2}^{1} xz \, dy \, dx = 0 \\ I_{yz} &= \int_{-1}^{1} \int_{x^2}^{1} yz \, dy \, dx = 0 \end{aligned}$$

Exercise 2.15. The position of the points (x_i, y_i, z_i) relative to the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is

$$(\hat{x}_i, \hat{y}_i, \hat{z}_i) = (x_i, y_i, z_i) - (\bar{x}, \bar{y}, \bar{z}) = (x_i - \bar{x}, y_i - \bar{y}, z_i - \bar{z})$$

Consider the I_{xx} inertia tensor component,

$$\begin{aligned} I_{xx} &= \textstyle\sum_{i=1}^{p} m_i(y_i^2 + z_i^2) \\ &= \textstyle\sum_{i=1}^{p} m_i[(\hat{y}_i + \bar{y})^2 + (\hat{z}_i + \bar{z})^2] \\ &= \textstyle\sum_{i=1}^{p} m_i(\hat{y}_i^2 + \hat{z}_i^2) + \sum_{i=1}^{p} m_i(2\hat{y}_i\bar{y} + \bar{y}^2 + 2\hat{z}_i\bar{z} + \bar{z}^2) \\ &= \textstyle\sum_{i=1}^{p} m_i(\hat{y}_i^2 + \hat{z}_i^2) + 2\bar{y}\sum_{i=1}^{p} m_i\hat{y}_i + 2\bar{z}\sum_{i=1}^{p} m_i\hat{z}_i + (\bar{y}^2 + \bar{z}^2)\sum_{i=1}^{p} m_i \\ &= \bar{I}_{xx} + 0 + 0 + m(\bar{y}^2 + \bar{z}^2) \end{aligned}$$

The middle two terms are zero because they include the first moments about the center of mass. For example, $0 = \sum_{i=1}^{p} m_i\hat{y}_i = \sum_{i=1}^{p} m_i(y_i - \bar{y})$.

Similarly,

$$\begin{aligned} I_{xy} &= \textstyle\sum_{i=1}^{p} m_i x_i y_i \\ &= \textstyle\sum_{i=1}^{p} m_i(\hat{x}_i + \bar{x})(\hat{y}_i + \bar{y}) \\ &= \textstyle\sum_{i=1}^{p} \hat{x}_i\hat{y}_i + \sum_{i=1}^{p} m_i(\bar{x}\hat{y}_i + \bar{y}\hat{x}_i + \bar{x}\bar{y}) \\ &= \textstyle\sum_{i=1}^{p} \hat{x}_i\hat{y}_i + \bar{x}\sum_{i=1}^{p} m_i\hat{y}_i + \bar{y}\sum_{i=1}^{p} m_i\hat{x}_i + \bar{x}\bar{y}\sum_{i=1}^{p} m_i \\ &= \bar{I}_{xy} + 0 + 0 + m\bar{x}\bar{y} \end{aligned}$$

Once again, the middle two terms are zero because they include the first moments about the center of mass.

The other inertia tensor components are handled in the same manner.

Exercise 2.17. The force is $\mathbf{F} = (1, 1, 1)$. The velocity of the particle is

$$\mathbf{v}(t) = (-r\omega \sin(\omega t), r\omega \cos(\omega t), 0)$$

The work done on the interval $t \in [0, T]$ is

$$\begin{aligned} W(T) &= \int_0^T (1, 1, 1) \cdot (-r\omega \sin(\omega t), r\omega \cos(\omega t), 0)\, dt \\ &= \int_0^T r\omega(\cos(\omega t) - \sin(\omega t))\, dt \\ &= r(\cos(\omega t) + \sin(\omega t))|_0^T \\ &= r(\cos(\omega T) + \sin(\omega T) - 1) \end{aligned}$$

In particular, at time $T = 2\pi\omega$, $W(2\pi\omega) = 0$. The force field is conservative, so the net work done is zero units for the particle to travel around the circle once and end where it started.

The net work is a maximum when $W'(T) = 0$. The derivative is

$$W'(T) = r\omega(-\sin(\omega T) + \cos(\omega t))$$

This is zero when $\tan(\omega T) = 1$, in which case ωT is $\pi/4 + k\pi$ for any integer k. Half these values lead to a minimum, the other half to a maximum. In particular, $T = \pi/(4\omega)$ leads to the maximum

$$W(\pi/(4\omega)) = r(\sqrt{2} - 1)$$

Now consider a force $\mathbf{F} = (t, t, 1)$. The work done on the interval $t \in [0, T]$ is

$$\begin{aligned} W(T) &= \int_0^T (t, t, 1) \cdot (-r\omega \sin(\omega t), r\omega \cos(\omega t), 0)\, dt \\ &= \int_0^T r\omega t(\cos(\omega t) - \sin(\omega t))\, dt \\ &= \tfrac{r}{\omega}\left(-1 + \cos(\omega T) - \sin(\omega T)) + T\omega(\cos(\omega T) + \sin(\omega T))\right) \end{aligned}$$

The integration may be performed using the method of integration by parts, looked up in a table of integrals, or computed symbolically (Mathematica for example).

The derivative is

$$W'(T) = r\omega T(\cos(\omega T) - \sin(\omega T))$$

and is zero when $\omega T = \pi/4 + k\pi$ for any integer k. At such values when $k = 2n$ is even, you have *local maxima*,

$$W(\pi/4 + 2n\pi) = r\omega(\pi/4 + 2n\pi)\sqrt{2}$$

But as n increases, so does the net work $W(\pi/4+2n\pi)$. This makes sense since the magnitude of $\mathbf{F}$ is $\sqrt{2t^2 + 1}$ which always increases over time. The force keeps getting stronger, so the net work keeps increasing.

3 Rigid Body Motion

Exercise 3.1. Multiply equation (3.2) by the denominator on the right-hand side,

$$e\rho = r(1 + e\cos\theta) = r + ex$$

where we have used $x = r\cos\theta$. Square both sides,

$$e^2\rho^2 = r^2 + 2exr + e^2x^2$$

and solve for

$$2exr = e^2(\rho^2 - x^2) - r^2$$

In polar coordinates we know that $r = \sqrt{x^2 + y^2}$. We need to eliminate the square root on the left, so square both sides once again,

$$4e^2x^2r^2 = [e^2(\rho^2 - x^2) - r^2]^2 = [e^2(\rho^2 - x^2)]^2 + r^4 - 2e^2(\rho^2 - x^2)r^2$$

Subtracting the left-hand side from both sides of the equation and perform the following steps,

$$\begin{aligned}
0 &= [e^2(\rho^2 - x^2)]^2 + r^4 - 2e^2(\rho^2 + x^2)r^2 \\
&= [e^2(\rho^2 + x^2) - 2e^2x^2]^2 + r^4 - 2e^2(\rho^2 + x^2)r^2 \\
&= [e^2(\rho^2 + x^2)]^2 - 4e^4x^2(\rho^2 + x^2) + 4e^4x^4 + r^4 - 2e^2(\rho^2 + x^2)r^2 \\
&= [e^2(\rho^2 + x^2)]^2 - 4e^4\rho^2x^2 + r^4 - 2e^2(\rho^2 + x^2)r^2 \\
&= [e^2(\rho^2 + x^2) - r^2]^2 - 4e^4\rho^2x^2 \\
4e^4\rho^2x^2 &= [e^2(\rho^2 + x^2) - r^2]^2
\end{aligned}$$

Take the square root of both sides,

$$\pm 2e^2\rho x = e^2(\rho^2 + x^2) - r^2$$

where the plus-or-minus sign indicates we have two possible solutions. Moving all variable terms to the left-hand side,

$$(1 - e^2)x^2 \pm 2e^2\rho x + y^2 = r^2 - e^2x^2 \pm 2e^2\rho x = e^2\rho^2$$

Now divide by $1 - e^2$ to obtain

$$x^2 \pm 2\frac{e^2\rho}{1 - e^2}\,x + \frac{1}{1 - e^2}\,y^2 = \frac{e^2\rho^2}{1 - e^2}$$

and complete the square

$$\left(x \pm \frac{e^2\rho}{1 - e^2}\right)^2 + \frac{1}{1 - e^2}\,y^2 = \frac{e^2\rho^2}{1 - e^2} + \frac{e^4\rho^2}{(1 - e^2)^2} = \frac{e^2\rho^2}{(1 - e^2)^2}$$

Recall that the book defines $a = e\rho/(1 - e^2)$ and $b = a\sqrt{1 - e^2}$, in which case $a^2/b^2 = 1/(1 - e^2)$. Also define $c = ae = e^2\rho/(1 - e^2)$. The last displayed equation becomes

$$(x \pm c)^2 + \frac{a^2}{b^2}\,y^2 = a^2$$

or finally

$$\frac{(x \pm c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the standard formula for an axis-aligned ellipse whose center is $(\pm c, 0)$, whose major axis length is $a > 0$, and whose minor axis length is $b > 0$.

In standard analytical geometry text books, the number c is introduced as the distance from the center of the ellipse to a focal point. In our example, this implies the origin $(0, 0)$ is a focal point of the ellipse. The other focal point is located at $(\pm 2c, 0)$.

Exercise 3.3. Kepler's Laws led us to the conclusion that the Earth and the Sun lie in a plane that is spanned by the position $\mathbf{r}$ of the Earth relative to the Sun and the velocity $\mathbf{v}$ of the Earth relative to the Sun. A normal vector to the plane of motion was shown to be the constant vector $\mathbf{c}_0 = \mathbf{r} \times \mathbf{v}$. In polar coordinates, $\mathbf{r} = r\mathbf{R}$ where r is the distance from the Sun to the Earth and $\mathbf{R}$ is a unit-length vector pointing from the Sun to the Earth. The distance r and angle θ are functions of time, say $r(t)$ and $\theta(t)$. The initial data is $r_0 = r(0)$, $\dot{r}_0 = \dot{r}(0)$, $\theta_0 = \theta(0)$, and $\dot{\theta}_0 = \dot{\theta}(0)$. The orbit of the Earth around the Sun in polar coordinates is $(r(t), \theta(t))$ for $t \geq 0$ and is uniquely determined for the specified initial data. The goal of the exercise is to show how to compute the orbit.

Item 1. Equation (3.2) is the representation of acceleration in polar coordinates (r, θ),

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{R} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{P}$$

where $\mathbf{R} = (\cos\theta, \sin\theta, 0)$ and $\mathbf{P} = (-\sin\theta, \cos\theta, 0)$. Notice that I have added the third component of 0 to indicate that as a system in 3D (cylindrical coordinates), the plane of motion of the Earth is $z = 0$. Equation (3.1) tells us that

$$\mathbf{a} = \dot{\mathbf{v}} = -\frac{GM}{r^3}\mathbf{r} = -\frac{GM}{r^2}\mathbf{R} = \left(-\frac{GM}{r^2}\right)\mathbf{R} + (0)\mathbf{P}$$

Equating the coefficients of $\mathbf{R}$ and $\mathbf{P}$ in the last two displayed equations,

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}, \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

Item 2. Multiply $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ by r to obtain

$$0 = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = \frac{d}{dt}\left(r^2\dot{\theta}\right)$$

Since the time derivative of $r^2\dot{\theta}$ is zero, the quantity is constant with respect to time. The mass M is also constant, so the angular momentum $\alpha = Mr^2\dot{\theta}$ is a constant. In particular, the constant is determined by the initial time, $\alpha = Mr_0^2\dot{\theta}_0$. Thus,

$$\dot{\theta}(t) = \frac{\alpha}{Mr^2} = \frac{r_0^2\dot{\theta}_0}{r^2} \tag{1}$$

If $\dot{\theta}_0 > 0$, then $\dot{\theta}(t) > 0$ for all time. For notations sake, let the angle be written as a function of time, $\theta = f(t)$. The fact that the derivative of θ with respect to time is always positive, $f(t)$ is a strictly increasing function of time. From calculus we know that such functions are invertible. The inverse is $t = f^{-1}(\theta)$. The distance is a function of time, $r(t)$. We can substitute in the expression for time to obtain

$$r(t) = r\left(f^{-1}(\theta)\right)$$

in which case r is also a function of θ.

Item 3. Using the first differential equation in Item 1 and replacing equation (1) in it,

$$-\frac{GM}{r^2} = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - r\left(\frac{\alpha}{Mr^2}\right)^2 = \ddot{r} - \frac{\alpha^2}{M^2r^3} \tag{2}$$

The right-hand side of the equation has a singularity at $r = 0$ which can cause problems when computing with a numerical differential equation solver.

Item 4. The potential energy as a function of polar angle θ is $V(\theta) = -GM/r$ where r is a function of θ. The derivative of V with respect to θ is denoted $V'(\theta)$. The initial conditions for potential energy are denoted $V_0 = V(\theta_0)$ and $V_0' = V'(\theta_0)$.

Item 4(a). Apply a time derivative to $r = -GM/V$,

$$\begin{aligned}
\dot{r} &= \tfrac{d}{dt}\left(-\tfrac{GM}{V}\right) \\
&= \tfrac{GM}{V^2}\tfrac{dV}{dt} && \text{by the power rule} \\
&= \tfrac{GM}{V^2}\tfrac{dV}{d\theta}\tfrac{d\theta}{dt} && \text{by the chain rule} \\
&= \tfrac{GM}{V^2}V'\dot{\theta} \\
&= \tfrac{GM}{(-GM/r)^2}V'\left(\tfrac{\alpha}{Mr^2}\right) \\
&= \tfrac{\alpha V'}{GM^2}
\end{aligned}$$

Item 4(b). Apply a time derivative to $\dot{r}$ from Item 4(a),

$$\begin{aligned}
\ddot{r} &= \tfrac{d\dot{r}}{dt} \\
&= \tfrac{\alpha}{GM^2}\tfrac{dV'}{dt} \\
&= \tfrac{\alpha}{GM^2}\tfrac{dV'}{d\theta}\tfrac{d\theta}{dt} && \text{by the chain rule} \\
&= \tfrac{\alpha}{GM^2}V''\left(\tfrac{\alpha}{Mr^2}\right) \\
&= \tfrac{\alpha^2 V''}{GM^3 r^2} \\
&= \tfrac{\alpha^2 V''}{GM^3(-GM/V)^2} \\
&= \tfrac{\alpha^2 V'' V^2}{G^3 M^5}
\end{aligned}$$

Item 4(c). Substitute $\ddot{r}$ from Item 4(b) into equation (2),

$$\begin{aligned}
-\tfrac{GM}{r^2} &= \ddot{r} - \tfrac{\alpha^2}{M^2 r^3} \\
-\tfrac{GM}{(-GM/V)^2} &= \tfrac{\alpha^2 V'' V^2}{G^3 M^5} - \tfrac{\alpha^2}{M^2(-GM/V)^3} \\
-\tfrac{V^2}{GM} &= \tfrac{\alpha^2 V'' V^2}{G^3 M^5} + \tfrac{\alpha^2 V^3}{G^3 M^5} \\
-\tfrac{G^2 M^4}{\alpha^2} &= V'' + V
\end{aligned}$$

The left-hand side is a constant. This is a nonhomogeneous, second-order, linear differential equation with constant coefficients. The characteristic polynomial is $\lambda^2 + 1 = 0$ and has roots $\lambda = \pm i$. This means the two linearly independent solutions to the homogeneous equation are $\sin(\theta)$ and $\cos(\theta)$. The general homogeneous solution is a linear combination of these,

$$V_h = c_0 \sin\theta + c_1 \cos\theta$$

where constants c_0 and c_1 are determined by initial data. Observe that $V_h'' + V_h = 0$. A particular solution to the nonhomogeneous equation is

$$V_p = -\frac{G^2 M^4}{\alpha^2}$$

Clearly, $V_p'' + V_p = 0 - G^2M^4/\alpha^2$, so V_p is a solution. The general solution to the differential equation is

$$V(\theta) = V_h + V_p = c_0 \sin\theta + c_1 \cos\theta - \frac{G^2M^4}{\alpha^2} \tag{3}$$

Item 4(d). The derivative of V in equation (3) is

$$V'(\theta) = c_0 \cos\theta - c_1 \sin\theta$$

Evaluating $V(\theta)$ and $V'(\theta)$ at the initial angle θ_0, we have two linear equations in the two unknown coefficients,

$$c_0 \sin\theta_0 + c_1 \cos\theta_0 = V_0 + \frac{G^2M^4}{\alpha^2}, \quad c_0 \cos\theta_0 - c_1 \sin\theta = V_0'$$

The solution is

$$c_0 = \left(V_0 + \frac{G^2M^4}{\alpha^2}\right)\sin\theta_0 + V_0' \cos\theta_0, \quad c_1 = \left(V_0 + \frac{G^2M^4}{\alpha^2}\right)\cos\theta_0 - V_0' \sin\theta_0$$

Finally, let us replace V_0 and V_0' by quantities involving the initial polar data r_0, $\dot{r}_0$, θ_0, and $\dot{\theta}_0$. First,

$$V_0 = V(\theta_0) = \frac{GM}{r_0}$$

Second, in Item 4(a) we showed that $\dot{r} = \alpha V'/(GM^2)$. At the initial time,

$$V_0' = \frac{GM^2\dot{r}_0}{\alpha}$$

Third, we had shown that $\alpha = Mr_0^2\dot{\theta}_0$. Combining these we have,

$$\begin{aligned} c_0 &= \left(V_0 + \tfrac{G^2M^4}{\alpha^2}\right)\sin\theta_0 + V_0'\cos\theta_0 \\ &= \left(\tfrac{GM}{r_0} + \tfrac{G^2M^4}{(Mr_0^2\dot{\theta}_0)^2}\right)\sin\theta_0 + \left(\tfrac{GM^2\dot{r}_0}{Mr_0^2\dot{\theta}_0}\right)\cos\theta_0 \\ &= \left(\tfrac{GM}{r_0} + \tfrac{G^2M^2}{r_0^4\dot{\theta}_0^2}\right)\sin\theta_0 + \left(\tfrac{GM\dot{r}_0}{r_0^2\dot{\theta}_0}\right)\cos\theta_0 \end{aligned} \tag{4}$$

Similarly,

$$c_1 = \left(\tfrac{GM}{r_0} + \tfrac{G^2M^2}{r_0^4\dot{\theta}_0^2}\right)\cos\theta_0 - \left(\tfrac{GM\dot{r}_0}{r_0^2\dot{\theta}_0}\right)\sin\theta_0 \tag{5}$$

Item 5. Conservation of momentum is represented by $\alpha = Mr^2\dot{\theta}$ where α is a constant. The definition of potential energy is $V = -GM/r$. Substitute this into the potential equation and solve for

$$\dot{\theta} = \frac{\alpha}{Mr^2} = \frac{\alpha}{M(-GM/V)^2} = \frac{\alpha V^2}{G^2M^3} \tag{6}$$

Item 6. In equation (6) substitute V from equation (3),

$$\dot{\theta} = \frac{\alpha}{G^2M^3}\left(c_0\sin\theta + c_1\cos\theta - \frac{G^2M^4}{\alpha^2}\right)^2, \quad \theta(0) = \theta_0 \tag{7}$$

The constants c_0 and c_1 are evaluated from equations (4) and (5).

Equation (7) is numerically solved in the application `KeplerPolarForm` on the CDROM. The application draws the elliptical path of motion of the Earth for user-specified inputs G, M, r_0, $\dot{r}_0$, θ_0, and $\dot{\theta}_0$. Once $\theta(t)$ is known for each t, equation (3.2) is used to compute $r(\theta) = e\rho/(1 + e\cos\theta)$ where $e = \gamma_1/(GM)$ and $\rho = \gamma_0^2/\gamma_1$.

We had shown that $\mathbf{c}_0 = \mathbf{r} \times \mathbf{v}$ is a constant for all time. We also showed that $\mathbf{v} \times \mathbf{c}_0 = GM\mathbf{R} + \mathbf{c}_1$ for a constant vector $\mathbf{c}_1$. The lengths of the constant vectors are $\gamma_0 = |\mathbf{c}_0|$ and $\gamma_1 = |\mathbf{c}_1|$. The constant vectors themselves can be computed at the initial time. By definition,

$$\mathbf{r} = r\mathbf{R}$$

From equation (2.12) the velocity is

$$\mathbf{v} = \dot{r}\mathbf{R} + r\dot{\theta}\mathbf{P}$$

The cross product is

$$\mathbf{r} \times \mathbf{v} = (r\mathbf{R}) \times (\dot{r}\mathbf{R} + r\dot{\theta}\mathbf{P}) = r^2\dot{\theta}\boldsymbol{k}$$

where $\boldsymbol{k} = (0, 0, 1)$. Consequently, at time zero,

$$\mathbf{c}_0 = r_0^2\dot{\theta}_0\boldsymbol{k}, \quad \gamma_0 = r_0^2\left|\dot{\theta}_0\right|$$

Furthermore,

$$\begin{aligned} \mathbf{v} \times \mathbf{c}_0 &= (\dot{r}\mathbf{R} + r\dot{\theta}\mathbf{P}) \times (r^2\dot{\theta})\boldsymbol{k}) \\ &= (r^2\dot{r}\dot{\theta})\mathbf{R} \times \boldsymbol{k} + (r^3\dot{\theta}^2)\mathbf{P} \times \boldsymbol{k} \\ &= -(r^2\dot{r}\dot{\theta})\mathbf{P} + (r^3\dot{\theta}^2)\mathbf{R} \end{aligned}$$

Consequently, at time zero,

$$\mathbf{c}_1 = (r_0^3\dot{\theta}_0^2)\mathbf{R}_0 - (r_0^2\dot{r}_0\dot{\theta}_0)\mathbf{P}_0 - GM\mathbf{R}_0 = (r_0^3\dot{\theta}_0^2 - GM)\mathbf{R}_0 - (r_0^2\dot{r}_0\dot{\theta}_0)\mathbf{P}_0$$

where $\mathbf{R}_0 = (\cos\theta_0, \sin\theta_0, 0)$ and $\mathbf{P}_0 = (-\sin\theta_0, \cos\theta_0, 0)$. The length is

$$\gamma_1 = \sqrt{(r_0^3\dot{\theta}_0^2 - GM)^2 + (r_0^2\dot{r}_0\dot{\theta}_0)^2}$$

The application is also designed to verify the equation we established for the period of the orbit,

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

where a is the major axis length of the ellipse. Recall from the book that $2a = 2\rho e/(1 - e^2)$ where The minor axis length is $2b = 2a\sqrt{1 - e^2}$. All of a, b, and T are computed in the application.

Exercise 3.5. Using the construction on page 97, the approximation for the integral is

$$T \doteq \int_0^{\pi/6-\varepsilon} \frac{d\psi}{\sqrt{\cos(\psi) - \cos(\pi/6)}} + \sqrt{\frac{2}{\cos(\pi/6)}} \left(\frac{\pi}{2} - \sin^{-1}\left(1 - \frac{\varepsilon \cos(\pi/6)}{\sin(\pi/6)}\right)\right)$$

for some small $\varepsilon > 0$. My implementation for approximating this is

```
#include "Wm5Integrate1.h"
#include "Wm5Math.h"
using namespace Wm5;

int main ()
{
    // integral term on [0,pi/6-e] (uses Simpson's rule for integration)
    double epsilon = 1e-08;
    double initialAngle = Mathd::PI/6.0;
    double amin = 0.0, amax = initialAngle - epsilon;
    double fmin = 1.0/Mathd::Sqrt(Mathd::Cos(amin) - initialAngle);
    double fmax = 1.0/Mathd::Sqrt(Mathd::Cos(amax) - initialAngle);
    double integral = fmin + fmax;
    const int imax = 1024;
    double h = (amax - amin)/(double)imax;
    for (int i = 1; i < imax; i++)
    {
        double angle = amin + i*h;
        double arg = Mathd::Cos(angle) - initialAngle;
        // assert: arg > 0.0
        double f = Mathd::InvSqrt(arg);
        if (i & 1)
        {
            integral += 4.0*f;
        }
        else
        {
            integral += 2.0*f;
        }
    }
    integral *= h/3.0;

    // remainder term on [pi/6-e,pi/6]
    double cs = Mathd::Cos(initialAngle);
    double sn = Mathd::Sin(initialAngle);
    double remainder = Mathd::Sqrt(2.0/cs)*(Mathd::HALF_PI -
        Mathd::ASin(1.0 - epsilon*cs/sn));

    double pendulumTime = integral + remainder;
    return 0;
}
```

The value of `pendulumTime` is 0.800275. The value of `remainder` is 0.000282.

Exercise 3.7. The position vector is $\mathbf{x} = (q, q^2, q^3)$ where. As a function of time, the initial position is $(1, 1, 1)$. The initial velocity is $(0, 0, 0)$ since the object is "released". First, let us set up the equations of motion. The q-derivative is $d\mathbf{x}/dq = (1, 2q, 3q^2)$. The t-derivative is $\dot{\mathbf{x}} = (d\mathbf{x}/dq)\dot{q}$. The kinetic energy is

$$T(q, \dot{q}) = \frac{m}{2}\,|\dot{\mathbf{x}}| = \frac{m}{2}(1 + 4q^2 + 9q^4)\dot{q}^2$$

The various derivatives and the generalized force are

$$\begin{aligned}
\tfrac{\partial T}{\partial q} &= 2m(2q + 9q^3)\dot{q}^2 \\
\tfrac{\partial T}{\partial \dot{q}} &= m(1 + 4q^2 + 9q^4)\dot{q} \\
\tfrac{d}{dt}\left(\tfrac{\partial T}{\partial \dot{q}}\right) &= m(1 + 4q^2 + 9q^4)\ddot{q} + 4m(2q + 9q^3)\dot{q}^2 \\
F_q &= -mg(0, 0, 1) \cdot (1, 2q, 3q^2) = -3mgq^2
\end{aligned}$$

The Lagrangian equation of motion is

$$\begin{aligned}
&\tfrac{d}{dt}\left(\tfrac{\partial T}{\partial \dot{q}}\right) - \tfrac{\partial T}{\partial q} = F_q \\
&m(1 + 4q^2 + 9q^4)\ddot{q} + 4m(2q + 9q^3)\dot{q}^2 - 2m(2q + 9q^3)\dot{q}^2 = -3mgq^2
\end{aligned}$$

or

$$\ddot{q} = -\frac{3gq^2 + 2q(2 + 9q^2)\dot{q}^2}{1 + 4q^2 + 9q^4}, \;\; q(0) = 1, \;\; \dot{q}(0) = 0$$

By assumption, $g > 0$. Notice that the right-hand side is always negative. This means that $\dot{q}$ is a decreasing function of time. Since the initial velocity is zero, it is the case that $\dot{q}(0) = 0$. Since $\dot{q}$ is decreasing, it must be that $\dot{q}(t) < 0$ for all times $t > 0$. This in turn implies that $q(t)$ is a decreasing function. This agrees with your intuition. Only gravity acts on the particle, so it must slide *down* the spiral curve, in which case q is decreasing.

The problem now is to determine the time $T > 0$ for which $q(T) = 0$. (This does require selecting a gravitational constant g.) At this time the position is $\mathbf{x}(T) = \mathbf{0}$. It is not possible to obtain a closed form formula for $q(t)$ from the second-order differential equation that defines it. Your only hope of *estimating* the time T is by a numerical method. An implementation of this is in the directory `BeadSlide` on the CDROM. I choose gravity $g = 1$, mass $m = 0.1$, a time step $h = 0.001$, and initial conditions $q(0) = 1$ and $\dot{q}(0) = 0$. I used a Runge-Kutta 4th-order solver and iterated 2500 times. The positions are constructed at each time. Two entries from the output file are

```
time       q                dot(q)       position (x,y,z)
2.240      +0.00017622      -1.41421347  +0.0002  +0.0000  +0.0000
2.241      -0.00123799      -1.41420923  -0.0012  +0.0000  -0.0000
```

The value of q reaches zero somewhere between 2.24 and 2.241. Notice that the position is approximately the zero vector.

Exercise 3.9. We already have the spring force

$$\mathbf{F} = -c(x_1, x_2, 0)$$

for a constant $c > 0$. Let the viscous friction be modeled by

$$\mathbf{G} = -a(\dot{x}_1, \dot{x}_2, 0)$$

for a constant $a > 0$. The total external force is $\mathbf{F} + \mathbf{G}$. The generalized forces are

$$F_{x_1} = (\mathbf{F} + \mathbf{G}) \cdot \frac{\partial \mathbf{X}}{\partial x_1} = (\mathbf{F} + \mathbf{G}) \cdot (1, 0, 0) = -cx_1 - a\dot{x}_1$$

and

$$F_{x_2} = (\mathbf{F} + \mathbf{G}) \cdot \frac{\partial \mathbf{X}}{\partial x_2} = (\mathbf{F} + \mathbf{G}) \cdot (0, 1, 0) = -cx_2 - a\dot{x}_2$$

The Lagrangian equations of motion are

$$\begin{aligned} m\ddot{x}_1 + a\dot{x}_1 + cx_1 = 0, \;\; x_1(0) = p_1, \dot{x}_1(0) = v_1 \\ m\ddot{x}_2 + a\dot{x}_2 + cx_2 = 0, \;\; x_2(0) = p_2, \dot{x}_2(0) = v_2 \end{aligned}$$

The equations are the same second-order linear differential equation with constant coefficients, so we will compute a solution $x(t)$ that represents either $x_1(t)$ or $x_2(t)$ depending on the initial conditions. The characteristic polynomial is $mr^2 + ar + = 0$, so the roots are

$$r = \frac{-a \pm \sqrt{a^2 - 4mc}}{2m}$$

Both roots are negative real-valued numbers (when $a^2 - 4mc \geq 0$) or are complex-valued with negative real parts (when $a^2 - 4mc < 0$).

Case 1. Let the roots be real-valued and named $r_1 = (-a + \sqrt{a^2 - 4mc})/(2m)$ and $r_2 = (-a - \sqrt{a^2 - 4mc})/(2m)$. The general solution to the differential equation is

$$x(t) = K_1 \exp(r_1 t) + K_2 \exp(r_2 t)$$

where the constants K_1 and K_2 are computed from the initial data for the differential equations. Regardless of their values, the fact that $r_1 < 0$ and $r_2 < 0$ guarantees that

$$\lim_{t \to \infty} x(t) = 0$$

Thus, the ball reaches the origin after an infinite amount of time. Well, this is only a physical model, so for all practical purposes, $x(t)$ becomes effectively zero after some amount of time. The larger the value of a, the sooner the ball gets (nearly) to the origin.

Case 2. Let the roots be complex-valued with negative real parts, say $r_1 = \alpha + \beta i$ and $r_2 = \alpha - \beta i$ where $\alpha = -a/(2m)$ and $\beta = \sqrt{4mc - a^2}/(2m)$. The general solution to the differential equation is

$$x(t) = \exp(\alpha t) \left[K_1 \cos(\beta t) + K_2 \sin(\beta t) \right]$$

The derivative is

$$\dot{x}(t) = \exp(\alpha t) \left[-\beta K_1 \sin(\beta t) + \beta K_2 \cos(\beta t) \right] + \alpha \exp(\alpha t) \left[K_1 \cos(\beta t) + K_2 \sin(\beta t) \right]$$

Let $x(0) = p$ and $\dot{x}(0) = v$. Replacing these in the equations for $x(t)$ and $\dot{x}(t)$ leads to $p = K_1$ and $v = \beta K_2 + \alpha K_1$. The solution is $K_1 = p$ and $K_2 = (v - \alpha p)/\beta$. Thus, the solutions are shown below with $i = 1$ or $i = 2$.

$$x_i(t) = \exp(\alpha t)\left[p_i \cos(\beta t) + \left(\frac{v_i - \alpha p_i}{\beta}\right)\sin(\beta t)\right]$$

The ball reaches the origin when $x_1(t) = 0$ and $x_2(t) = 0$. Since the exponential terms are not zero (for finite time), the implied conditions are

$$\begin{aligned} p_1 \cos(\beta t) + \left(\tfrac{v_1 - \alpha p_1}{\beta}\right)\sin(\beta t) &= 0 \\ p_2 \cos(\beta t) + \left(\tfrac{v_2 - \alpha p_2}{\beta}\right)\sin(\beta t) &= 0 \end{aligned}$$

Let $\mathbf{p} = (p_1, p_2)$ and $\mathbf{v} = (v_1, v_2)$. If $\mathbf{p}$ and $(\mathbf{v} - \alpha\mathbf{p})/\beta$ (that is, $\mathbf{p}$ and $\mathbf{v}$ are parallel, then the two displayed equations are really only one independent equation. Sovling the first we have,

$$t = \frac{1}{\beta}\tan^{-1}\left(\frac{-\beta p_1}{v_1 - \alpha p_1}\right)$$

The ball reaches the origin in finite time. Notice that physically what happens is that the velocity of the ball is initially pointing to the origin. The ball is (eventually) pulled directly to the origin.

If $\mathbf{p}$ and $(\mathbf{v} - \alpha\mathbf{p})/\beta$ are not parallel, then the two equations are of the form

$$(\cos(\beta t), \sin(\beta t)) \cdot \left(p_i, \frac{v_i - \alpha p_i}{\beta}\right) = 0$$

Thus,

$$\left(p_i, \frac{v_i - \alpha p_i}{\beta}\right) = \lambda_i(-\sin(\beta t), \cos(\beta t))$$

for some scalars λ_i. However this implies that $\mathbf{p}$ and $(\mathbf{v} - \alpha\mathbf{p})/beta$ are parallel, a contradiction to our initial assumption. Consequently, both equations cannot be simultaneously zero and the ball can never reach the origin in *finite time*. It does in infinite time since $\lim_{t\to\infty}\exp(\alpha t) = 0$ because $\alpha < 0$.

Exercise 3.11. Let $(x_1, x_2) = (v_1 t, v_2 t)$ for some constants v_1 and v_2. The derivatives are $(\dot{x}_1, \dot{x}_2) = (v_1, v_2)$ and $(\ddot{x}_1, \ddot{x}_2) = (0, 0)$. Replacing these in the first equation of motion,

$$\begin{aligned} 0 &= \ddot{x}_1 + \tfrac{4x_1}{a_1}\left(\tfrac{x_1\ddot{x}_1+\dot{x}_1^2}{a_1^2} + \tfrac{x_2\ddot{x}_2+\dot{x}_2^2}{a_2^2}\right) - \tfrac{2gx_1}{a_1^2} \\ &= 0 + \tfrac{4v_1 t}{a_1^2}\left(\tfrac{v_1^2}{a_1^2} + \tfrac{v_2^2}{a_2^2}\right) - \tfrac{2gv_1 t}{a_1^2} \end{aligned}$$

A few algebraic steps will show that

$$\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2} = \frac{g}{2}$$

The same condition is derived by substitution of the proposed solution into the second equation of motion. The only way $(x_1, x_2) = (v_1 t, v_2 t)$ is a solution is if this condition is satisfied. If the condition is not satisfied, then the path of motion in the (x_1, x_2) plane is not a straight line.

Exercise 3.13. To construct the Lagrangian equations of motion, we need to construct a height function for the chute, a function $x_3 = h(x_1, x_2)$. The independent variables x_1 and x_2 are what the Lagrangian equations are based on.

Consider a cylinder whose axis is the parametric line $\mathbf{P} + t\mathbf{D}$ where $\mathbf{P}$ is a point on the line and $\mathbf{D}$ is a unit-length direction for the line. If R is the radius of the cylinder, the general quadratic equation that defines the cylinder is

$$(\mathbf{X} - \mathbf{P})^{\mathrm{T}} \left(I - \mathbf{D}\mathbf{D}^{\mathrm{T}} \right) (\mathbf{X} - \mathbf{P}) = R^2$$

where I is the identity matrix and the superscript T denotes the transpose operation.

I hear you ask "Where did this come from?" The point $\mathbf{P}$ may be used as the origin of a coordinate system with coordinate axis directions $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{D}$. The directions are all unit-length and mutually perpendicular. Any point $\mathbf{X}$ may be written in the coordinate system as

$$\mathbf{X} = \mathbf{P} + y_1\mathbf{U} + y_2\mathbf{V} + y_3\mathbf{D}$$

with $y_1 = \mathbf{U} \cdot (\mathbf{X} - \mathbf{P})$, $y_2 = \mathbf{V} \cdot (\mathbf{X} - \mathbf{P})$, and $y_3 = \mathbf{D} \cdot (\mathbf{X} - \mathbf{P})$. For a point on the cylinder, the distance from the axis is R units. This distance is measured in the plane spanned by $\mathbf{U}$ and $\mathbf{V}$, so $y_1^2 + y_2^2 = R^2$ is required. Rearranging terms,

$$y_1\mathbf{U} + y_2\mathbf{V} = (\mathbf{X} - \mathbf{P}) - (\mathbf{D} \cdot (\mathbf{X} - \mathbf{P}))\mathbf{D}$$

The squared lengths of the two sides of the equation must be equal. The left-hand side has squared length $y_1^2 + y_2^2 = R^2$. The right-hand side has squared length:

$$\begin{aligned}
R^2 &= \left|(\mathbf{X} - \mathbf{P}) - (\mathbf{D} \cdot (\mathbf{X} - \mathbf{P}))\mathbf{D}\right|^2 \\
&= (\mathbf{X} - \mathbf{P}) \cdot (\mathbf{X} - \mathbf{P}) - 2(\mathbf{D} \cdot (\mathbf{X} - \mathbf{P}))^2 + (\mathbf{D} \cdot (\mathbf{X} - \mathbf{P}))^2(\mathbf{D} \cdot \mathbf{D}) \\
&= (\mathbf{X} - \mathbf{P}) \cdot (\mathbf{X} - \mathbf{P}) - (\mathbf{D} \cdot (\mathbf{X} - \mathbf{P}))^2 \\
&= (\mathbf{X} - \mathbf{P})^{\mathrm{T}}(\mathbf{X} - \mathbf{P}) - ((\mathbf{X} - \mathbf{P})^{\mathrm{T}}\mathbf{D})(\mathbf{D}^{\mathrm{T}}(\mathbf{X} - \mathbf{P})) \\
&= (\mathbf{X} - \mathbf{P})^{\mathrm{T}} I (\mathbf{X} - \mathbf{P}) - (\mathbf{X} - \mathbf{P})^{\mathrm{T}}(\mathbf{D}\mathbf{D}^{\mathrm{T}})(\mathbf{X} - \mathbf{P}) \\
&= (\mathbf{X} - \mathbf{P})^{\mathrm{T}} \left(I - \mathbf{D}\mathbf{D}^{\mathrm{T}} \right) (\mathbf{X} - \mathbf{P})
\end{aligned}$$

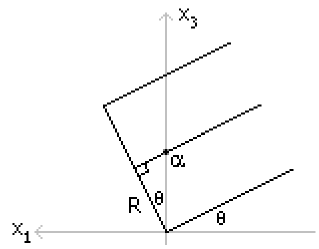
In our particular problem, the cylinder axis direction is obtained from basic algebra and trigonometry. The axis lies in the plane $x_2 = 0$. Using part (b) of the figure that is associated with the exercise, if x_3 increases by H units, then x_1 decreases by $L \cos\theta$ units. The side of the triangle opposite to θ has length H, the hypotenuse has length L, so the adjacent side has length $\sqrt{L^2 - H^2}$. Using the fact that cosine is adjacent over hypotenuse, $\cos\theta = \sqrt{L^2 - H^2}/L$. Note also that $\sin\theta = H/L$. A unit-length direction for the cylinder is therefore

$$\mathbf{D} = (-\cos\theta, 0, \sin\theta)$$

Now to compute a point on the cylinder axis. The next figure shows a point

$$\mathbf{P} = (0, 0, \alpha)$$

on the axis.



Clearly, $R = \alpha \cos\theta$, so $\alpha = R/\cos\theta$. Let $\mathbf{D} = (d_1, 0, d_3)$, $\mathbf{P} = (0, 0, p_3)$, and $\mathbf{X} = (x_1, x_2, x_3)$. We will use the fact that $d_1^2 + d_3^2 = 1$ several times. Expanding the cylinder equation,

$$
\begin{aligned}
0 &= (\mathbf{X} - \mathbf{P})^{\mathrm{T}} \left(I - \mathbf{D}\mathbf{D}^{\mathrm{T}} \right) (\mathbf{X} - \mathbf{P}) - R^2 \\
&= x_1^2 + x_2^2 + (x_3 - p_3)^2 - (d_1 x_1 + d_3(x_3 - p_3))^2 \\
&= (1 - d_3)^2 (x_3 - p_3)^2 - 2 d_1 d_3 x_1 (x_3 - p_3) + (1 - d_1^2) x_1^2 + x_2^2 - R^2 \\
&= d_1^2 (x_3 - p_3)^2 - 2 d_1 d_3 x_1 (x_3 - p_3) + d_3^2 x_1^2 + x_2^2 - R^2 \\
&= [d_1 (x_3 - p_3) - d_3 x_1]^2 + x_2^2 - R^2
\end{aligned}
$$

Thus,

$$d_1 (x_3 - p_3) - d_3 x_1 = \pm\sqrt{R^2 - x_2^2}$$

The choice of sign on the right-hand side is based on knowing that the point $(x_1, x_2, x_3) = (0, 0, 0)$ is on the cylinder. Substituting in this point, we have

$$(\cos\theta)(R/\cos\theta) = -d_1 p_3 = \pm R$$

For this to be equal, we need the plus sign on the right. Consequently,

$$d_1 (x_3 - p_3) - d_3 x_1 = \sqrt{R^2 - x_2^2}$$

defines the chute. Solving for x_3,

$$x_3 = p_3 + \frac{d_3 x_1 + \sqrt{R^2 - x_2^2}}{d_1} = \frac{LR - H x_1 - L\sqrt{R^2 - x_2^2}}{L \cos\theta}$$

The height function $h(x_1, x_2)$ is the right-hand side of this equation.

The exercise is to compute the Lagrangian equations of motion. The details are tedious, but let's look ahead to Exercise 3.14 where you are asked to construct the equations of motion for a height field in general. Using the formula constructed in that exercise, we need to substitute in the first- and second-order partial derivatives of $h(x_1, x_2)$. These are

$$h_{x_1} = \frac{-H}{L\cos\theta}, \quad h_{x_2} = \frac{x_2}{(R^2 - x_2^2)^{1/2} \cos\theta}, \quad h_{x_1 x_1} = h_{x_1 x_2} = 0, \quad h_{x_2 x_2} = \frac{R^2}{(R^2 - x_2^2)^{3/2} \cos\theta}$$

Substituting into the general equations of motion,

$$\left[\begin{array}{c} \ddot{x}_1 \\ \ddot{x}_2 \end{array} \right] = -\left(\frac{\frac{R^2 \dot{x}_2^2}{(R^2 - x_2^2)^{3/2} \cos\theta} + g}{1 + \frac{H^2}{L^2 \cos^2\theta} + \frac{x_2^2}{(R^2 - x_2^2)\cos^2\theta}} \right) \left[\begin{array}{c} \frac{-H}{L\cos\theta} \\ \frac{x_2}{(R^2 - x_2^2)^{1/2} \cos\theta} \end{array} \right]$$

Notice that $x_2(t) \equiv 0$ is a solution to the second equation of the system. That means if the ball starts in the center of the chute with no initial speed in the x_2 component, then it must remain in the center of the chute. The other equation of motion tells you how x_1 varies,

$$\ddot{x}_1 = \frac{g\left(\frac{H}{L\cos\theta}\right)}{1+\left(\frac{H}{L\cos\theta}\right)^2} = \frac{g\tan\theta}{1+\tan^2\theta} = \frac{g}{2}\sin(2\theta)$$

The right-hand side is a positive constant. The acceleration in the x_1 direction is always positive, so the speed increases as well as the position in that direction:

$$x_1(t) = x_1(0) + \dot{x}_1(0)t + (g\sin(2\theta)/4)t^2$$

Notice that as you increase the angle θ, the acceleration increases, as you expect.

If you add viscous friction, the differential equation for x_2 is still of the form $\ddot{x}_2 = Sx_2$ where S is a scalar function of various parameters, including g and any new terms introduced by the addition of viscous friction. The function $x_2(t) \equiv 0$ is still a solution, so if the ball starts in the center of the chute with no initial speed in the x_2 direction, it remains in the center of the chute.

Exercise 3.15. The angle between the line segment and vertical axis is $\theta = \pi/4$. The initial direction of the segment is $\mathbf{D}_0 = (1,0,1)/\sqrt{2}$. The line segment is rotated about the x_3-axis by an amount θt for time t. The rotation is towards the (x_2, x_3) plane. The direction of the segment at that time is

$$\mathbf{D}(t) = \mathrm{Rot}(\theta t, \boldsymbol{k})\mathbf{D}_0 = (\cos(\theta t), \sin(\theta t), 1)/\sqrt{2}$$

The position of the mass is initially at $(b, 0, b)$. At later time, the mass is on the line segment with direction $\mathbf{D}(t)$, so is

$$\mathbf{x}(t) = \lambda(t)\mathbf{D}(t)$$

where $0 \le \lambda(t) \le L$, L is the length of the line segment, and $\lambda(0) = bL/a$. The mass is constrained to lie on the line segment, so the parameter λ is the one degree of freedom in the system. To set up the problem using Lagrangian dynamics, we think of the position as a function of time t and parameter λ, say $\mathbf{x}(t, \lambda)$.

The velocity is

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t} + \dot{\lambda}\frac{\partial \mathbf{x}}{\partial \lambda} = \lambda\dot{\mathbf{D}}(t) + \dot{\lambda}\mathbf{D}(t)$$

where

$$\dot{\mathbf{D}}(t) = \theta(-\sin(\theta t), \cos(\theta t), 0)/\sqrt{2}$$

The kinetic energy is

$$T = \frac{m}{2}\left|\dot{\mathbf{x}}\right|^2 = \frac{m}{2}\left|\lambda\dot{\mathbf{D}} + \dot{\lambda}\mathbf{D}\right|^2 = \frac{m}{2}\left(\frac{1}{2}\theta^2\lambda^2 + \dot{\lambda}^2\right)$$

The relevant derivatives are

$$\frac{\partial T}{\partial \lambda} = \frac{m\theta^2}{2}\lambda, \quad \frac{\partial T}{\partial \dot{\lambda}} = m\dot{\lambda}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\lambda}}\right) = m\ddot{\lambda}$$

The force on the mass uses Hooke's Law. The rest length ℓ of the spring is the length of the line segment connecting $(c, 0, 0)$ to $(b, 0, b)$,

$$\ell = \sqrt{(b-c)^2 + b^2}$$

The spring constant is $K > 0$. The force $\mathbf{F}$ exerted on the mass is towards the fixed point $(c, 0, 0)$,

$$\mathbf{F} = -K(|\mathbf{x}(t) - c\boldsymbol{\imath}| - \ell)\frac{\mathbf{x}(t) - c\boldsymbol{\imath}}{|\mathbf{x}(t) - c\boldsymbol{\imath}|}$$

where $\boldsymbol{\imath} = (1, 0, 0)$. At time zero, $\mathbf{x}(0) = (b, 0, b)$, in which case $|\mathbf{x}(0) - c\boldsymbol{\imath}| = |(b - c, 0, b)| = \ell$ and $\mathbf{F}(0) = \mathbf{0}$. The spring is initially unstretched, as hypothesized. The generalized force is

$$\begin{aligned} F_\lambda &= \mathbf{F} \cdot \tfrac{\partial \mathbf{x}}{\partial \lambda} \\ &= -K(|\mathbf{x}(t) - c\boldsymbol{\imath}| - \ell)\tfrac{\mathbf{x}(t) - c\boldsymbol{\imath}}{|\mathbf{x}(t) - c\boldsymbol{\imath}|} \cdot \mathbf{D}(t) \\ &= -K\left(1 - \tfrac{\ell}{\sqrt{\lambda^2 - \sqrt{2}c\cos(\theta t)\lambda + c^2}}\right)(\lambda - c\cos(\theta t)/\sqrt{2}) \end{aligned}$$

The equation of motion is

$$m\ddot{\lambda} - \frac{m\theta^2}{2}\lambda = F_\lambda, \quad \lambda(0) = bL/a$$

Since this is a second-order equation, we also need to know $\dot{\lambda}(0)$ before we can solve this using numerical methods. The initial line segment on which the mass lies is initially at rest, so the mass has initial velocity $\dot{\mathbf{x}}(0) = \mathbf{0}$. Also true is $\dot{\lambda}(t) = \mathbf{D}(t) \cdot \dot{\mathbf{x}}(t)$, so

$$\dot{\lambda}(0) = \mathbf{D}(0) \cdot \dot{\mathbf{x}}(0) = (a, 0, a) \cdot (0, 0, 0) = 0$$

You are now ready to solve the equation numerically. The complexity of the differential equation does not lead to a closed form solution, so a numerical approach is your only hope.

Notice that the time the mass arrives in the (x_2, x_3) plane is determined by $\pi t/4 = \theta t = \pi/2$, so $t = 2$. You need to numerically solve for $d = \lambda(2)$.

In summary, you need to solve

$$\ddot{\lambda} = \frac{\pi^2}{32}\lambda - \frac{K}{m}\left(1 - \frac{\ell}{\sqrt{\lambda^2 - 2\lambda\cos(\pi t/4)c/sqrt2 + c^2}}\right)(\lambda - \cos(\pi t/4)c/\sqrt{2})$$

with $\lambda(0) = bL/a$ and $\dot{\lambda}(0) = 0$. The parameters you need to select in the solver are L, c, m, and K. The value $\ell = \sqrt{(b-c)^2 + c^2}$. You need to numerically solve for $d = \lambda(2)$. The same conversion is used as always to reduce the second-order equation to a first-order system, $\dot{\lambda} = u$, and $\dot{u} = \ddot{\lambda}$.

The following code uses the same pattern that occurs in the many physics applications on the CDROM. The file `PhysicsModule.h` contains

```
#ifndef PHYSICSMODULE_H
#define PHYSICSMODULE_H

#include "Wm5OdeSolver.h"
#include "Wm5Vector3.h"

class PhysicsModule
{
public:
    // Construction and destruction.
    PhysicsModule ();
    ~PhysicsModule ();

    // Initialize the differential equation solver.
    void Initialize (double time, double deltaTime, double lambda,
        double lambdaDot);

    double GetTime () const       { return mTime; }
    double GetDeltaTime () const  { return mDeltaTime; }
    double GetLambda () const     { return mState[0]; }
    double GetLambdaDot () const  { return mState[1]; }
    Wm5::Vector3d GetPosition () const;

    // Take a single step of the solver.
    void Update ();

    // System parameters.
    double Mass;            // m
```

```
    double SegmentLength;   // L, a = L/sqrt(2)
    double SpringEnd;       // c
    double SpringConstant;  // K

private:
    // state and auxiliary variables
    enum { PM_STATES = 2, PM_AUXS = 7 };
    double mTime, mDeltaTime, mState[PM_STATES], mAux[PM_AUXS];

    // ODE solver (specific solver assigned in the cpp file)
    Wm5::OdeSolverd* mSolver;
    static void OdeFunction (double time, const double* state,
        void* data, double* output);
};

#endif
```

The file PhysicsModule.cpp contains

```
#include "PhysicsModule.h"
#include "Wm5OdeRungeKutta4.h"
#include "Wm5Math.h"
using namespace Wm5;

//----------------------------------------------------------------------------
PhysicsModule::PhysicsModule ()
{
    Mass = 0.0;
    SegmentLength = 0.0;
    SpringEnd = 0.0;
    SpringConstant = 0.0;
    memset(mState,0, PM_STATES*sizeof(double));
    memset(mAux,0, PM_AUXS*sizeof(double));
    mSolver = 0;
}
//----------------------------------------------------------------------------
PhysicsModule::~PhysicsModule ()
{
    delete mSolver;
}
//----------------------------------------------------------------------------
void PhysicsModule::Initialize (double time, double deltaTime,
    double lambda, double lambdaDot)
{
    mTime = 0.0;
    mDeltaTime = deltaTime;

    // State variables.
    mState[0] = lambda;
    mState[1] = lambdaDot;

    // Auxiliary variables.
```

```
    mAux[0] = Mathd::Sqrt(0.5);  // 1/sqrt(2)
    mAux[1] = 0.25*Mathd::PI;  // pi/4
    mAux[2] = Mathd::PI*Mathd::PI/32.0;  // pi^2/32
    mAux[3] = SpringEnd*mAux[0];  // c/sqrt(2)
    mAux[4] = SpringEnd*SpringEnd; // c^2
    mAux[5] = SpringConstant/Mass; // K/m
    mAux[6] = (GetPosition()-SpringEnd*Vector3d::UNIT_X).Length();  // ell

    // RK4 differential equation solver.
    delete mSolver;
    mSolver = new OdeRungeKutta4d(PM_STATES,mDeltaTime,OdeFunction, mAux);
}
//----------------------------------------------------------------------------
Vector3d PhysicsModule::GetPosition () const
{
    double lambdaDivSqrt2 = mState[0]*mAux[0];
    double thetaTimesT = mAux[1]*mTime;
    return Vector3d(
        lambdaDivSqrt2*Mathd::Cos(thetaTimesT),
        lambdaDivSqrt2*Mathd::Sin(thetaTimesT),
        lambdaDivSqrt2);
}
//----------------------------------------------------------------------------
void PhysicsModule::Update ()
{
    // Take a single step in the ODE solver.
    mSolver->Update(mTime, mState, mTime, mState);
}
//----------------------------------------------------------------------------
void PhysicsModule::OdeFunction (double time, const double* state,
    void* data, double* output)
{
    double* aux = (double*)data;

    // lambda function
    output[0] = state[1];

    // dot(lambda) function

    // c*cos(pi*t/4)/sqrt(2)
    double tmp0 = aux[3]*Mathd::Cos(aux[0]*time);

    // lambda - c*cos(pi*t/4)/sqrt(2)
    double tmp1 = state[0] - tmp0;

    // lambda - 2*c*cos(pi*t/4)/sqrt(2)
    double tmp2 = state[0] - 2.0*tmp0;

    // lambda*(lambda - 2*c*cos(pi*t/4)/sqrt(2)) + c^2
    double tmp3 = state[0]*tmp2 + aux[4];

    // 1 - ell/sqrt(lambda*(lambda - 2*c*cos(pi*t/4)/sqrt(2) + c^2))
```

```
    double tmp4 = 1.0 - aux[6]*Mathd::InvSqrt(tmp3);

    output[1] = aux[2]*state[0] - aux[5]*tmp4*tmp1;
}
//----------------------------------------------------------------------------
```

The main program is in a file `TestExample3p15.cpp`

```
#include "PhysicsModule.h"
using namespace Wm5;

int main ()
{
    PhysicsModule module;
    module.Mass = 1.0;
    module.SegmentLength = 4.0;
    module.SpringEnd = 1.0;
    module.SpringConstant = 10.0;

    double time = 0.0;
    double deltaTime = 0.01;
    double lambda = 0.5*module.SegmentLength; // start at midpoint
    double lambdaDot = 0.0;
    module.Initialize(time, deltaTime, lambda, lambdaDot);

    std::ofstream outFile("solution.txt");
    Vector3d pos = module.GetPosition();
    outFile << "i = 0 ";
    outFile << "lambda = " << module.GetLambda() << ' ';
    outFile << "x1 = " << pos.X() << ' ';
    outFile << "x2 = " << pos.Y() << ' ';
    outFile << "x3 = " << pos.Z() << std::endl;

    double finalTime = 2.0;
    const int imax = (int)(finalTime/deltaTime + 0.5);
    for (int i = 1; i <= imax; i++)
    {
        module.Update();
        pos = module.GetPosition();
        outFile << "i = " << i << ' ';
        outFile << "lambda = " << module.GetLambda() << ' ';
        outFile << "x1 = " << pos.X() << ' ';
        outFile << "x2 = " << pos.Y() << ' ';
        outFile << "x3 = " << pos.Z() << std::endl;
    }
    return 0;
}
```

The first and last lines of the output file `solution.txt` are shown below (with minor formatting changes)

```
 i =   0  lambda = 2        x1 = 1.41421        x2 = 0          x3 = 1.41421
```

```
i = 200  lambda = 1.15214  x1 = -8.54596e-016  x2 = 0.814683  x3 = 0.814683
```

Notice that x_1 is effectively zero, so the final position of the mass is in the (x_2, x_3) plane, as desired. The value we wanted to compute is $d = 0.814683$. Notice that λ has decreased, so the mass has slipped down the line segment towards the origin.

Exercise 3.17. The Foucault pendulum equations of motion were established using Newton's Law of Motion. A Lagrangian approach can be used instead. From pages 92 through 94, the position of the mass m is

$$\mathbf{r}(t) = L\mathbf{R}$$

where L is the length of the pendulum rod and where the coordinate axes are

$$\begin{aligned}
\mathbf{P} &= (-\sin\theta, \cos\theta, 0) \\
\mathbf{Q} &= (-\cos\theta\cos\phi, -\sin\theta\cos\phi, \sin\phi) \\
\mathbf{R} &= (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)
\end{aligned}$$

The velocity is

$$\mathbf{v}(t) = \frac{D\mathbf{r}}{Dt} = L\left[(\dot\theta\sin\phi)\mathbf{P} - \dot\phi\mathbf{Q}\right]$$

The kinetic energy is

$$T = \frac{m}{2}|\mathbf{v}(t)|^2 = \frac{mL^2}{2}(\dot\theta^2\sin^2\phi + \dot\phi^2)$$

The relevant derivatives are

$$\begin{array}{l}
\frac{\partial T}{\partial\theta} = 0, \quad \frac{\partial T}{\partial\phi} = mL^2\dot\theta^2\sin\phi\cos\phi, \quad \frac{\partial T}{\partial\dot\theta} = mL^2\dot\theta\sin^2\phi, \quad \frac{\partial T}{\partial\dot\phi} = mL^2\dot\phi \\
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) = mL^2(\ddot\theta\sin^2\phi + 2\dot\theta\dot\phi\sin\phi\cos\phi), \quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) = mL^2\ddot\phi
\end{array}$$

If F_θ and F_ϕ are the generalized forces, then the equations of motion are

$$\begin{aligned}
&mL^2(\ddot\theta\sin^2\phi + 2\dot\theta\dot\phi\sin\phi\cos\phi) = F_\theta \\
&mL^2(\ddot\phi - \dot\theta^2\sin\phi\cos\phi) = F_\phi
\end{aligned}$$

The force on the mass itself is

$$\mathbf{F} = m\frac{D^2\mathbf{r}}{Dt} = m\left(-2\mathbf{w}\times\frac{D\mathbf{r}}{Dt} + g\boldsymbol{k} - \tau\mathbf{R}\right)$$

The generalized force with respect to θ is

$$F_\theta = \mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial\theta} = \mathbf{F}\cdot L\frac{\partial\mathbf{R}}{\partial\theta} = (mL\sin\phi)\frac{D^2\mathbf{r}}{Dt}\cdot\mathbf{P} = (mL\sin\phi)[2L\omega\dot\phi(-\cos\lambda\sin\theta\sin\phi + \sin\lambda\cos\phi)]$$

The last equality comes from the displayed equation on page 93 that immediately precedes equation (3.7). The generalized force with respect to ϕ is

$$F_\phi = \mathbf{F}\cdot\frac{\partial\mathbf{r}}{\partial\phi} = \mathbf{F}\cdot L\frac{\partial\mathbf{R}}{\partial\phi} = (-mL)\frac{D^2\mathbf{r}}{Dt}\cdot\mathbf{Q} = (-mL)[2L\omega\dot\theta\sin\phi(-\cos\lambda\sin\theta\sin\phi + \sin\lambda\cos\phi) + g\sin\phi]$$

The last equality comes from the second displayed equation on page 94.

Substituting these into our earlier equations, canceling the common term $mL^2\sin\phi$ from the first equation, and canceling the common term mL^2 from the second equation, we have

$$\begin{aligned}
&\ddot\theta\sin\phi + 2\dot\theta\dot\phi\cos\phi = 2\omega\dot\phi(-\cos\lambda\sin\theta\sin\phi + \sin\lambda\cos\phi) \\
&\ddot\phi - \dot\theta^2\sin\phi\cos\phi = -2\omega\dot\theta\sin\phi(-\cos\lambda\sin\theta\sin\phi + \sin\lambda\cos\phi) - \tfrac{g}{L}\sin\phi
\end{aligned}$$

which are exactly the equations we derived using Newton's Laws.

The problem is now modified so that the point mass is replaced by a solid cone of mass m, height h, and radius r. The idea is similar to that shown in Figure 3.12. The value L measures the distance from the joint of the pendulum to the center of mass of the cone. The kinetic energy has an additional term, and is now

$$T = \frac{m}{2}|\mathbf{v}(t)|^2 + \frac{1}{2}\left(\mathbf{w}^{\mathrm{T}} J \mathbf{w}\right)$$

The second term involves the angular velocity $\mathbf{w}$ and the world inertia tensor J of the cone. The cone axis is a principal direction, and is in the direction of $\mathbf{R}$. The vectors $\mathbf{P}$ and $\mathbf{Q}$ serve as two other principal directions, based on the symmetry of the cone.

Consider the standard cone of height h and radius r whose circular base is on the xy-plane, whose axis is the z-axis, and whose vertex is at $(0, 0, h)$. Let's assume a mass density of 1. The mass is just the volume of the cone,

$$m = \frac{\pi r^2 h}{3}$$

By symmetry, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ has components $\bar{x} = \bar{y} = 0$. The $\bar{z}$ component must be chosen so that half the volume lies above $z = \bar{z}$. The region above $z = \bar{z}$ is itself a cone of radius $r(1 - \bar{z}/h)$ and height $h - \bar{z}$. The volume is $\pi r^2(1 - \bar{z}/h)^2(h - \bar{z})/3$. The ratio of this volume to that of the full cone must be

$$\frac{1}{2} = \frac{\pi r^2(1 - \bar{z}/h)^2(h - \bar{z})/3}{\pi r^2 h/3} = (1 - \bar{z}/h)^3$$

which implies

$$\bar{z} = h\left(1 - 2^{-1/3}\right)$$

Consequently,

$$L = h - \bar{z} = h/2^{1/3} \doteq 0.7937h$$

The inertia tensor component I_{xx} is constructed below where V is the region occupied by the cone, $\rho = r(1 - z/h)$

$$\begin{array}{rcl}
I_{xx} & = & \int_V y^2 + z^2\, dx\, dy\, dz \\
& = & \int_0^h \int_{-\rho}^{\rho} \int_{-\sqrt{\rho^2-y^2}}^{\sqrt{\rho^2-y^2}} y^2 + z^2\, dx\, dy\, dz \\
& = & \int_0^h \int_{-\rho}^{\rho} 2(y^2 + z^2)\sqrt{\rho^2 - y^2}\, dy\, dz \\
& = & \int_0^h \left(\int_{-\rho}^{\rho} 2y^2\sqrt{\rho^2 - y^2}\, dy\right) + z^2\left(\int_{-\rho}^{\rho} \sqrt{\rho^2 - y^2}\, dy\right)\, dz \\
& = & \int_0^h (\pi\rho^4/4 + \pi\rho^2 z^2)\, dz \\
& = & \pi\int_0^h \left(r^4(1 - z/h)^4/4 + z^2(1 - z/h)^2\right)\, dz \\
& = & \frac{\pi r^2 h(2h^2+3r^2)}{60} \\
& = & m\,\frac{2h^2+3r^2}{20}
\end{array}$$

By symmetry of the cone with respect to x and y, or by a similar integration,

$$I_{yy} = I_{xx} = m\,\frac{2h^2 + 3r^2}{20}$$

Also by symmetry,

$$I_{xy} = I_{xz} = I_{yz} = 0$$

The last tensor component is obtained by integration, but using cylindrical coordinates $s = \sqrt{x^2 + y^2}$, θ, and z,

$$\begin{aligned} I_{zz} &= \textstyle\int_V x^2 + y^2 \, dx \, dy \, dz \\ &= \textstyle\int_0^h \int_0^{2\pi} \int_0^{r(1-z/h)} s^3 \, ds \, d\theta \, dz \\ &= \tfrac{\pi r^4 h}{10} \\ &= m \, \tfrac{3r^2}{10} \end{aligned}$$

The inertia tensor in world coordinates is

$$J = \begin{bmatrix} \mathbf{P} & \mathbf{Q} & \mathbf{R} \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{xx} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{\mathrm{T}} \\ \mathbf{Q}^{\mathrm{T}} \\ \mathbf{R}^{\mathrm{T}} \end{bmatrix} = I_{xx}(\mathbf{P}\mathbf{P}^{\mathrm{T}} + \mathbf{Q}\mathbf{Q}^{\mathrm{T}}) + I_{zz}\mathbf{R}\mathbf{R}^{\mathrm{T}}$$

In the book we had $\mathbf{w} = \omega[(\cos\lambda)\boldsymbol{\jmath} - (\sin\lambda)\boldsymbol{k}]$. Consequently,

$$\begin{aligned} \mathbf{w}^{\mathrm{T}} J \mathbf{w} &= I_{xx}\left((\mathbf{P}\cdot\mathbf{w})^2 + (\mathbf{Q}\cdot\mathbf{w})^2\right) + I_{zz}(\mathbf{R}\cdot\mathbf{w})^2 \\ &= I_{xx}\omega^2[(\cos\lambda\cos\theta)^2 + (\cos\lambda\sin\theta\cos\phi + \sin\lambda\sin\phi)^2] + I_{zz}\omega^2(\cos\lambda\sin\theta\sin\phi - \sin\lambda\cos\phi)^2 \end{aligned}$$

The derivatives for T that we had in the case of a single point mass must be appended with the various derivatives of $\mathbf{w}^{\mathrm{T}} J \mathbf{w}/2$. As you can see, this will be a tedious chore, but tractable. If you want to actually code this, I recommend using a symbolic mathematics program to generate these derivatives, and the resulting C code.

Exercise 3.19. The presence of the y-axis label in Figure 3.13 is misleading, but the text that goes with the figure correctly states the situation. The value y_3 is the distance from the ceiling to the center of the pulley. The value y_1 is the *additional distance* needed to get from the pulley center to the mass m_1. That is, y_1 is *not* the distance from the ceiling to the mass. Similarly, y_2 is the additional distance needed to get from the pulley center to the mass m_2.

Figure 3.15 has a different intent, just to help clarify. The value y_1 is the distance from the ceiling to the center of the large pulley. The value y_2 is the distance from the ceiling to the small pulley center. The value y_3 is the distance from the ceiling to the mass m_3.

First, notice that y_1 does not vary in this problem. The values y_2 and y_3 vary. The angles θ_1 and θ_2 for the pulleys, measured similar to that of Figure 3.13, also vary. Any change in θ_1 is directly related to a change in $y_2 - y_1$. Similarly, any change in θ_2 is directly related to a change in $y_3 - y_2$. Thus,

$$R_1\dot{\theta}_1 = \dot{y}_2 - \dot{y}_1 = \dot{y}_2, \ \ R_2\dot{\theta}_2 = \dot{y}_3 - \dot{y}_2$$

Our system has only two degrees of freedom, y_2 and y_3.

The kinetic energy is

$$T = \frac{m_2}{2}\dot{y}_2^2 + \frac{I_1}{2}\dot{\theta}_1^2 + \frac{m_3}{2}\dot{y}_3^2 + \frac{I_2}{2}\dot{\theta}_2^2 = \frac{m_2}{2}\dot{y}_2^2 + \frac{I_1}{2R_1^2}\dot{y}_2^2 + \frac{m_3}{2}\dot{y}_3^2 + \frac{I_2}{2R_2^2}(\dot{y}_3 - \dot{y}_2)^2$$

The relevant derivatives are

$$\begin{aligned}
\frac{\partial T}{\partial y_2} &= 0 \\
\frac{\partial T}{\partial y_3} &= 0 \\
\frac{\partial T}{\partial \dot{y}_2} &= m_2\dot{y}_2 + \frac{I_1}{R_1^2}\dot{y}_2 + \frac{I_2}{R_2^2}(\dot{y}_2 - \dot{y}_3) \\
\frac{\partial T}{\partial \dot{y}_3} &= m_3\dot{y}_3 + \frac{I_2}{R_2^2}(\dot{y}_3 - \dot{y}_2) \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_2}\right) &= m_2\ddot{y}_2 + \frac{I_1}{R_1^2}\ddot{y}_2 + \frac{I_2}{R_2^2}(\ddot{y}_2 - \ddot{y}_3) \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_3}\right) &= m_3\ddot{y}_3 + \frac{I_2}{R_2^2}(\ddot{y}_3 - \ddot{y}_2)
\end{aligned}$$

We need to calculate the generalized forces on the objects. Let S_1 denote the length of the spring: $S_1 > L_1$ indicates stretching, $S_1 < L_1$ indicates compression. The force on the small pulley has two components, a gravitational one and one due to the spring,

$$\mathbf{F}_2 = m_2 g\boldsymbol{\jmath} - c_1(S_1 - L_1)\boldsymbol{\jmath}$$

Observe that $S_1 - y_2 = K_1$, a constant. This is just a statement that the wire connecting the top of the spring to the center of the small pulley must be constant. If the small pulley moves downward, then the string is stretched, in which case S_1 increases. However, y_2 can also increase by exactly that the increase in S_1 since the wire is fixed length. The force on the small pulley is therefore

$$\mathbf{F}_2 = m_2 g\boldsymbol{\jmath} - c_1(K_1 - L_1 + y_2)\boldsymbol{\jmath}$$

Let S_2 denote the length of the spring: $S_2 > L_2$ indicates stretching, $S_2 < L_2$ indicates compression. The force on the point mass is

$$\mathbf{F}_3 = m_3 g\boldsymbol{\jmath} - c_2(S_2 - L_2)\boldsymbol{\jmath}$$

Observe that $S_2 - y_3 + 2y_2 = K_2$, a constant. This is also a statement about conservation of length. For a fixed y_2, if the point mass moves downward, say $y_3 \to y_3 + \Delta$, then the string must be stretched accordingly, say $S_2 \to S_2 + \Delta$. Then

$$(S_2 + \Delta) - (y_3 + \Delta) + 2y_2 = S_2 - y_3 + 2y_2 = K_2$$

Now if y_2 increases, say $y_2 \to y_2 + \Delta$, then the small pulley moves downward. The string must be compressed simultaneously while y_3 increases, say $S_2 \to S_2 - \Delta$ and $y_3 \to y_3 + \Delta$. Then

$$(S_2 - \Delta) - (y_3 + \Delta) + 2(y_2 + \Delta) = S_2 - y_3 + 2y_2 = K_2$$

The force on the point mass is therefore

$$\mathbf{F}_3 = m_3 g \boldsymbol{\jmath} - c_2(K_2 - L_2 + y_3 - 2y_2)\boldsymbol{\jmath}$$

The generalized forces are

$$F_{y_2} = m_2 g - c_1(K_1 - L_1 + y_2)$$

and

$$F_{y_3} = m_3 g - c_2(K_2 - L_2 + y_3 - 2y_2)$$

The equations of motion are

$$\begin{aligned} m_2\ddot{y}_2 + \tfrac{I_1}{R_1^2}\ddot{y}_2 + \tfrac{I_2}{R_2^2}(\ddot{y}_2 - \ddot{y}_3) &= m_2 g - c_1(K_1 - L_1 + y_2) \\ m_3\ddot{y}_3 + \tfrac{I_2}{R_2^2}(\ddot{y}_3 - \ddot{y}_2) &= m_3 g - c_2(K_2 - L_2 + y_3 - 2y_2) \end{aligned}$$

Exercise 3.21. The equations of motion for the system were shown to be

$$\ddot{\theta} = \frac{F_\theta}{(mL^2 + \mu_2)\sin^2\phi}, \quad \ddot{\psi} = \frac{-F_\theta \cos\phi}{(mL^2 + \mu_2)\sin^2\phi}$$

The generalized force for θ was shown to be

$$F_\theta = \mathbf{F} \cdot L(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$

If the force is gravitational, $\mathbf{F} = -mg\boldsymbol{k}$, then $F_\theta = 0$ for all time. Thus,

$$\ddot{\theta} = 0, \quad \ddot{\psi} = 0$$

This says the angular accelerations of the pipe and disk are zero. Integrating, the values $\dot{\theta}(t) = \dot{\theta}_0$ and $\dot{\psi}(t) = \dot{\psi}(0)$ for all time. That is, the angular speeds are constant.

Exercise 3.23. The potential energy for gravity is

$$V_{\text{gravity}} = mgr\sin\theta$$

and the potential energy for the spring is

$$V_{\text{spring}} = \frac{c}{2}\left(\left(\sqrt{s^2(\cos\theta-1)^2+(h+s\sin\theta)^2}-\ell\right)^2-(h-\ell)^2\right)$$

The potential energy is

$$V = V_{\text{gravity}} + V_{\text{spring}}$$

the kinetic energy is

$$T = \frac{m}{2}r^2\dot{\theta}^2$$

and the Lagrangian is

$$L = T - V$$

The equation of motion is

$$\begin{aligned}
0 &= \tfrac{d}{dt}\left(\tfrac{\partial L}{\partial\dot{\theta}}\right) - \tfrac{\partial L}{\partial\theta} \\
&= mr^2\ddot{\theta} + \tfrac{\partial V}{\partial\theta} \\
&= mr^2\ddot{\theta} + mgr\cos\theta + c\left(1 - \tfrac{\ell}{\sqrt{s^2(\cos\theta-1)^2+(h+s\sin\theta)^2}}\right)\left(s^2(1-\cos\theta)\sin\theta + s(h+s\sin\theta)\cos\theta\right)
\end{aligned}$$

For equilibrium at $\theta = 0$, we need all the terms, minus the $\ddot{\theta}$ term, to evaluate to zero when θ is zero. This condition is

$$mgr + cs(h-\ell) = 0$$

just as it was in the book.

Exercise 3.25. The kinetic energy is

$$T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) + \frac{m_3}{2}(\dot{x}_3^2 + \dot{y}_3^2)$$

but only three variables are freely chosen. Once again we choose the angles to be the degrees of freedom. The construction in Example 3.15 provides equations you can use here, but with a few more added to them. The position of the third mass is

$$\begin{aligned} x_3 &= r_1 \sin\theta_1 + r_2 \sin\theta_2 + r_3 \sin\theta_3 \\ y_3 &= h - r_1 \cos\theta_1 - r_2 \cos\theta_2 - r_3 \cos\theta_3 \end{aligned}$$

and the derivatives are

$$\begin{aligned} \dot{x}_3 &= r_1\dot{\theta}_1 \cos\theta_1 + r_2\dot{\theta}_2 \cos\theta_2 + r_3\dot{\theta}_3 \cos\theta_3 \\ \dot{y}_3 &= r_1\dot{\theta}_1 \sin\theta_1 + r_2\dot{\theta}_2 \sin\theta_2 + r_3\dot{\theta}_3 \sin\theta_3 \end{aligned}$$

The kinetic energy reduces to

$$\begin{aligned} T &= \tfrac{m_1}{2}\left((r_1\dot{\theta}_1\cos\theta_1)^2 + (r_1\dot{\theta}_1\sin\theta_1)^2\right) + \\ &\quad \tfrac{m_2}{2}\left((r_1\dot{\theta}_1\cos\theta_1 + r_2\dot{\theta}_2\cos\theta_2)^2 + (r_1\dot{\theta}_1\sin\theta_1 + r_2\dot{\theta}_2\sin\theta_2)^2\right) + \\ &\quad \tfrac{m_3}{2}\left((r_1\dot{\theta}_1\cos\theta_1 + r_2\dot{\theta}_2\cos\theta_2 + r_3\dot{\theta}_3\cos\theta_3)^2 + (r_1\dot{\theta}_1\sin\theta_1 + r_2\dot{\theta}_2\sin\theta_2 + r_3\dot{\theta}_3\sin\theta_3)^2\right) \\ &= \tfrac{m_1}{2}\left(r_1^2\dot{\theta}_1^2\right) + \\ &\quad \tfrac{m_2}{2}\left(r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2 + 2r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) +\right) \\ &\quad \tfrac{m_3}{2}\left(r_1^2\dot{\theta}_1^2 + r_2^2\dot{\theta}_2^2 + r_3^2\dot{\theta}_3^2 + 2r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + 2r_1r_3\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3) + 2r_2r_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)\right) \end{aligned}$$

The potential energy is

$$V = -m_1 g(h - y_1) - m_2 g(h - y_2) - m_3 g(h - y_3)$$

The Lagrangian is $L = T - V$ and the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0,\ \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0,\ \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} = 0$$

I will leave it to you to crank out those derivatives. If you had a chain of n masses connected by rigid rods, the details are even more horrendous. It is possible, however, to figure out the pattern and write source code to handle n masses. Or if you prefer, generate source code using a symbolic mathematics package.

Exercise 3.27. The three degrees of freedom are r, θ, and ϕ. We chose $\dot{\theta}_0 = 0$, which led to $\theta(t) = \theta(0) = \theta_0$ for all time (θ is constant). You need to specify the initial angle θ_0. You also need to specify the initial data $r_0 = r(0)$, $\dot{r}_0 = \dot{r}(0)$, $\phi_0 = \phi(0)$, and $\dot{\phi}_0 = \dot{\phi}(0)$. The first differential equation to solve is

$$\ddot{r} = \frac{\beta^2}{r^3} - \frac{\gamma}{r^2}$$

where

$$\beta = r_0^2\dot{\phi}_0, \quad \gamma = \frac{c_2}{c_1} = \frac{Gm_1m_2}{m_1m_2/(m_1+m_2)} = G(m_1+m_2)$$

The parameters G, m_1, and m_2 must be selected by the user. The second differential equation to solve is

$$\dot{\phi} = \frac{\beta}{r^2}$$

By defining $u = \dot{r}$, we have set up the problem as a system of three first-order equations,

$$\begin{array}{rcll} \dot{r} & = & u, & r(0) = r_0 \\ \dot{u} & = & \frac{\beta^2}{r^3} - \frac{\gamma}{r^2}, & u(0) = \dot{r}_0 \\ \dot{\phi} & = & \frac{\beta}{r^2}, & \phi(0) = \phi_0 \end{array}$$

The pattern that appears in the `PhysicsModule` files may be used here. The state variables are r, u, and ϕ. The public data members are m_1, m_2, and G. The initial conditions are r_0, $\dot{r}_0$, ϕ_0, and $\dot{\phi}_0$. The auxiliary variables are β and γ.

Exercise 3.29. The essence of the problem is discussed in Section 2.2.2. The position function is decomposed by using the center of mass as the origin, and the columns of an orientation matrix as the coordinate axes. For the thin rod,

$$\mathbf{r} = (x, y) + L(\cos\theta, \sin\theta)$$

where (x, y) is the center of mass. You may rewrite this in matrix form as

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} L \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + R(\theta) \begin{bmatrix} L \\ 0 \end{bmatrix}$$

with $L \in [-L_2, L_1]$. In "body space", the center of mass of the rod is located at the origin and oriented to be on the x-axis. The orientation matrix $R(\theta)$ rotates the rod about the origin. Then the center of mass in the world is added to the rod, the final result being in "world space".

A general curve can be defined in body space. The center of mass $\mathbf{c}$ is computed according to the integral methods on page 48 (curve in the plane) or page 53 (curve in space). The center of mass is subtracted from your initial curve description. You may or may not want to reorient the initial curve. In the end, you have a curve $\mathbf{b}(s)$ in body space. I have indicated the variable is arc length s, but the discussion on pages 48 and 53 show how you convert back to the curve parameter. The arc length parameter is in $[0, L]$ where L is the total length of the curve. In 2D the position is

$$\mathbf{r}(t, s) = \mathbf{c}(t) + R(\theta(t))\mathbf{b}(s)$$

The velocity is

$$\mathbf{v}(t, s) = \frac{\partial \mathbf{r}}{\partial t} = \dot{\mathbf{c}} + \dot{\theta}\frac{dR}{d\theta}\mathbf{b}(s)$$

where

$$\dot{\mathbf{c}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, \quad \frac{dR}{d\theta} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

Thus,

$$\mathbf{v} = \begin{bmatrix} \dot{x} - \dot{\theta}(b_0 \sin\theta + b_1 \cos\theta) \\ \dot{y} + \dot{\theta}(b_0 \cos\theta - b_1 \sin\theta) \end{bmatrix}$$

where $\mathbf{b} = (b_0, b_1)$.

Let the mass density be $\delta(s)$. The infinitesimal mass is $dm = \delta(s)ds$ and the total mass is

$$\mu_0 = \int_0^L dm = \int_0^L \delta(s)\, ds$$

The kinetic energy is

$$T(x, y, \theta) = \int_0^L \frac{dm}{2}|\mathbf{v}|^2 = \int_0^L \frac{1}{2}|\mathbf{v}|^2\, \delta(s)\, ds$$

But notice that

$$|\mathbf{v}|^2 = \dot{x}^2 + \dot{y}^2 + \dot{\theta}^2(b_0^2 + b_1^2)$$

so

$$T = \frac{\mu_0}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\mu_2}{2}\dot{\theta}^2$$

where

$$\mu_2 = \int_0^L |\mathbf{b}(s)|^2 \, \delta(s) \, ds$$

The only differences between this formula and the one for a thin rod are the values of μ_0 and μ_2. The set up for the Lagrangian equations of motion proceeds as usual, computing the relevant derivatives of the kinetic energy.

The generalized forces are

$$\begin{aligned}
F_x &= \textstyle\int_0^L \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial x} \, \delta(s) \, ds = \int_0^L \mathbf{F} \cdot \boldsymbol{\imath} \, \delta(s) \, ds \\
F_y &= \textstyle\int_0^L \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial y} \, \delta(s) \, ds = \int_0^L \mathbf{F} \cdot \boldsymbol{\jmath} \, \delta(s) \, ds \\
F_\theta &= \textstyle\int_0^L \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \, \delta(s) \, ds = \int_0^L \mathbf{F} \cdot \frac{dR}{d\theta} \mathbf{b} \, \delta(s) \, ds
\end{aligned}$$

The integrals are as complicated as for the thin rod, especially when rough friction is used. However, numerical integration techniques still apply.

Exercise 3.31. The application on the CDROM named `RoughPlaneSolidBox` is an implementation that uses rough friction. It does not incorporate static friction, but may be easily modifed to do so.

Exercise 3.33. The center of mass of the top is located at $\mathbf{r} = \ell\boldsymbol{\xi}_3$, as mentioned in Example 3.19. As in the example, we look at the system in body coordinates. The velocity of the center of mass is zero in body coordinates: $\boldsymbol{\xi}_3$ is unchanging in body coordinats since it is a body axis.

The contribution of the angular velocity $\mathbf{w}$ to the kinetic energy is

$$T = \frac{1}{2}\mathbf{w}^{\mathrm{T}}J\mathbf{w}$$

where $\mathbf{w}$ is the angular velocity in body coordinates and J is the inertia tensor in body coordinates. We know in body coordinates that the top is radially symmetric about the top axis, so its inertia tensor is the diagonal matrix

$$J = \mathrm{Diag}(\mu_1, \mu_1, \mu_3)$$

The angular velocity in body coordinates is provided in equation (3.48),

$$\mathbf{w}_{\mathrm{body}} = (\dot\phi\cos\psi + \dot\theta\sin\psi\sin\phi)\boldsymbol{\xi}_1 + (-\dot\phi\sin\phi + \dot\theta\cos\psi\sin\phi)\boldsymbol{\xi}_2 + (\dot\psi + \dot\theta\cos\phi)\boldsymbol{\xi}_3$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}\mathbf{w}^{\mathrm{T}}J\mathbf{w} \\ &= \tfrac{\mu_1}{2}(\dot\phi\cos\psi + \dot\theta\sin\psi\sin\phi)^2 + \tfrac{\mu_1}{2}(-\dot\phi\sin\phi + \dot\theta\cos\psi\sin\phi)^2 + \tfrac{\mu_3}{2}(\dot\psi + \dot\theta\cos\phi)^2 \\ &= \tfrac{\mu_1}{2}(\dot\phi^2 + \dot\theta^2\sin^2\phi) + \tfrac{\mu_3}{2}(\dot\psi + \dot\theta\cos\phi)^2 \end{aligned}$$

The relevant derivatives are

$$\frac{\partial T}{\partial\theta} = 0, \quad \frac{\partial T}{\partial\phi} = \mu_1\dot\theta^2\sin\phi\cos\phi + \mu_3(\dot\psi + \dot\theta\cos\phi)(-\dot\theta\sin\phi), \quad \frac{\partial T}{\partial\psi} = 0$$

and

$$\frac{\partial T}{\partial\dot\theta} = \mu_1\dot\theta\sin^2\phi + \mu_3(\dot\psi + \dot\theta\cos\phi)\cos\phi, \quad \frac{\partial T}{\partial\dot\phi} = \mu_1\dot\phi, \quad \frac{\partial T}{\partial\dot\psi} = \mu_3(\dot\psi + \dot\theta\cos\phi)$$

The Lagrangian equations of motion are

$$\begin{aligned} &\tfrac{d}{dt}\Big(\mu_1\dot\theta\sin^2\phi + \mu_3(\dot\psi + \dot\theta\cos\phi)\cos\phi\Big) = F_\theta \\ &\tfrac{d}{dt}\Big(\mu_1\dot\phi\Big) - \mu_1\dot\theta^2\sin\phi\cos\phi + \mu_3(\dot\psi + \dot\theta\cos\phi)(\dot\theta\sin\phi) = F_\phi \\ &\tfrac{d}{dt}\Big(\mu_3(\dot\psi + \dot\theta\cos\phi)\Big) = F_\psi \end{aligned}$$

where F_θ, F_ϕ, and F_ψ are the generalized forces. Specifically,

$$\begin{aligned} F_\theta &= (-mg\boldsymbol{\eta}_3)\cdot\tfrac{\partial\mathbf{r}}{\partial\theta} \\ &= (-mg\boldsymbol{\eta}_3)\cdot\ell\,[(\cos\theta\sin\phi)\boldsymbol{\eta}_1 + (\sin\theta\sin\phi)\boldsymbol{\eta}_2 + (0)\boldsymbol{\eta}_3] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} F_\phi &= (-mg\boldsymbol{\eta}_3)\cdot\tfrac{\partial\mathbf{r}}{\partial\phi} \\ &= (-mg\boldsymbol{\eta}_3)\cdot\ell\,[(\sin\theta\cos\phi)\boldsymbol{\eta}_1 + (-\cos\theta\cos\phi)\boldsymbol{\eta}_2 + (-\sin\phi)\boldsymbol{\eta}_3] \\ &= mg\ell\sin\phi \end{aligned}$$

and

$$\begin{aligned} F_\psi &= (-mg\boldsymbol{\eta}_3)\cdot\tfrac{\partial\mathbf{r}}{\partial\theta} \\ &= (-mg\boldsymbol{\eta}_3)\cdot\ell\left[(0)\boldsymbol{\eta}_1+(0)\boldsymbol{\eta}_2+(0)\boldsymbol{\eta}_3\right] \\ &= 0 \end{aligned}$$

The equations of motion are

$$\begin{aligned} &\tfrac{d}{dt}\left(\mu_1\dot\theta\sin^2\phi+\mu_3(\dot\psi+\dot\theta\cos\phi)\cos\phi\right)=0 \\ &\tfrac{d}{dt}\left(\mu_1\dot\phi\right)-\mu_1\dot\theta^2\sin\phi\cos\phi+\mu_3(\dot\psi+\dot\theta\cos\phi)(\dot\theta\sin\phi)=mg\ell\sin\phi \\ &\tfrac{d}{dt}\left(\mu_3(\dot\psi+\dot\theta\cos\phi)\right)=0 \end{aligned}$$

Observe that the last equation implies

$$\dot\psi+\dot\theta\cos\phi=c,$$

a constant, just as we had shown using the Eulerian approach. Replacing this in the first equation, and using the fact that the time derivative of the expression is zero,

$$\mu_1\dot\theta\sin^2\phi+c\mu_3\cos\phi=\gamma$$

where γ is a constant. Again, this is what we had shown in the Eulerian approach. The middle equation reduces to

$$\ddot\phi-\dot\theta^2\sin\phi\cos\phi+\frac{c\mu_3}{\mu_1}\dot\theta\sin\phi=\alpha\sin\phi$$

where $\alpha=mg\ell/\mu_1$.

Exercise 3.35. The tip of the top moves about the plane only because some force has been applied to the center of mass. The Euler equations of motion from Example 3.19 still apply for the motion of the top about its axis and for the motion of the top about the world vertical axis with origin at the tip of the top.

The motion of the tip (center of mass) may be handled separately. In this case, the tip is moved by $(x(t), y(t), z(t)) = (\alpha \sin(\lambda t), 0, 0)$.

If you like, modify the sample application `FreeTopFixedTip` to include the motion of the tip as listed here.

4 Deformable Bodies

No exercises.

5 Fluids and Gases

Exercise 5.1

The proof is

$$\begin{aligned}
(\mathbf{f} \cdot \nabla)\mathbf{x} &= \left(f_0 \tfrac{\partial}{\partial x_0} + f_1 \tfrac{\partial}{\partial x_1} + f_2 \tfrac{\partial}{\partial x_2}\right) \cdot (x_0, x_1, x_2) \\
&= f_0 \tfrac{\partial(x_0,x_1,x_2)}{\partial x_0} + f_1 \tfrac{\partial(x_0,x_1,x_2)}{\partial x_1} + f_2 \tfrac{\partial(x_0,x_1,x_2)}{\partial x_2} \\
&= f_0(1,0,0) + f_1(0,1,0) + f_2(0,0,1) \\
&= (f_0, f_1, f_2) \\
&= \mathbf{f}
\end{aligned}$$

Exercise 5.3

Using the quotient rule for differentiation,

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\left(\frac{x_i}{(x_0^2+x_1^2+x_2^2)^{3/2}}\right) &= \frac{(x_0^2+x_1^2+x_2^2)^{3/2}-3x_i^2(x_0^2+x_1^2+x_2^2)^{1/2}}{(x_0^2+x_1^2+x_2^2)^{6/2}} \\
&= \frac{x_0^2+x_1^2+x_2^2-3x_i^2}{(x_0^2+x_1^2+x_2^2)^{5/2}}
\end{aligned}
$$

Then

$$
\begin{aligned}
\nabla\cdot\left(\frac{\mathbf{X}}{|\mathbf{X}|^3}\right) &= \frac{\partial}{\partial x_0}\left(\frac{x_i}{(x_0^2+x_1^2+x_2^2)^{3/2}}\right)+\frac{\partial}{\partial x_1}\left(\frac{x_1}{(x_0^2+x_1^2+x_2^2)^{3/2}}\right)+\frac{\partial}{\partial x_2}\left(\frac{x_2}{(x_0^2+x_1^2+x_2^2)^{3/2}}\right) \\
&= \frac{(x_0^2+x_1^2+x_2^2-3x_0^2)+(x_0^2+x_1^2+x_2^2-3x_1^2)+(x_0^2+x_1^2+x_2^2-3x_2^2)}{(x_0^2+x_1^2+x_2^2)^{5/2}} \\
&= 0
\end{aligned}
$$

Exercise 5.5

Using tensor index notation, M is represented by m_{ij} and $\mathbf{f}$ is represented by f_j. The product $M\mathbf{f}$ is represented by $m_{ij}f_j$, where the repeated index j indicates you must sum over j. The index i is the only free index, which indicates that $m_{ij}f_j$ represents a vector. The divergence of this vector is

$$\nabla \cdot (M\mathbf{f}) = \frac{\partial}{\partial x_i}(m_{ij}f_j) = \left(m_{ij}\frac{\partial}{\partial x_i}\right) f_j$$

where the constants m_{ij} are moved outside the derivative (the typical rule of differentiation).

For a vector $\mathbf{g}$ represented by g_i, the product $\mathbf{g}^{\mathrm{T}}M$ is represented by $g_i m_{ij}$, where the repeated index i indicates you must sum over i. The index j is the only free index, which indicates that $g_i m_{ij}$ represents a vector. Scalar multiplication is commutative, so we may rewriting this expression as $m_{ij}g_i$. In vector notation, this also represents $M^{\mathrm{T}}\mathbf{g}$. In the previously displayed equation we have the expression

$$m_{ij}\frac{\partial}{\partial x_i}$$

which is of the same form as $m_{ij}g_i$. Thus, this expression represents $M^{\mathrm{T}}\nabla$, in which case

$$\nabla \cdot (M\mathbf{f}) = \left(m_{ij}\frac{\partial}{\partial x_i}\right) f_j = (M^{\mathrm{T}}\nabla) \cdot \mathbf{f}$$

Exercise 5.7

The proof is

$$\begin{aligned}
\nabla \times (\phi \mathbf{f}) &= \left(\tfrac{\partial(\phi f_2)}{\partial x_1} - \tfrac{\partial(\phi f_1)}{\partial x_2}, \tfrac{\partial(\phi f_0)}{\partial x_2} - \tfrac{\partial(\phi f_2)}{\partial x_0}, \tfrac{\partial(\phi f_1)}{\partial x_0} - \tfrac{\partial(\phi f_0)}{\partial x_1} \right) \\
&= \left(\phi \tfrac{\partial f_2}{\partial x_1} + \tfrac{\partial \phi}{\partial x_1} f_2 - \phi \tfrac{\partial f_1}{\partial x_2} - \tfrac{\partial \phi}{\partial x_2} f_1, \phi \tfrac{\partial f_0}{\partial x_2} + \tfrac{\partial \phi}{\partial x_2} f_0 - \phi \tfrac{\partial f_2}{\partial x_0} - \tfrac{\partial \phi}{\partial x_0} f_2, \phi \tfrac{\partial f_1}{\partial x_0} + \tfrac{\partial \phi}{\partial x_0} f_1 - \phi \tfrac{\partial f_0}{\partial x_1} - \tfrac{\partial \phi}{\partial x_1} f_0 \right) \\
&= \phi \left(\tfrac{\partial f_2}{\partial x_1} - \tfrac{\partial f_1}{\partial x_2}, \tfrac{\partial f_0}{\partial x_2} - \tfrac{\partial f_2}{\partial x_0}, \tfrac{\partial f_1}{\partial x_0} - \tfrac{\partial f_0}{\partial x_1} \right) + \left(\tfrac{\partial \phi}{\partial x_1} f_2 - \tfrac{\partial \phi}{\partial x_2} f_1, \tfrac{\partial \phi}{\partial x_2} f_0 - \tfrac{\partial \phi}{\partial x_0} f_2, \tfrac{\partial \phi}{\partial x_0} f_1 - \tfrac{\partial \phi}{\partial x_1} f_0 \right) \\
&= \phi \nabla \times \mathbf{f} + \nabla \phi \times \mathbf{f}
\end{aligned}$$

Exercise 5.9

The proof is

$$\begin{aligned}
\nabla \times (\nabla\phi) &= \nabla \times \left(\frac{\partial\phi}{\partial x_0}, \frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\right) \\
&= \left(\frac{\partial}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial}{\partial x_2}\frac{\partial\phi}{\partial x_1}, \frac{\partial}{\partial x_2}\frac{\partial\phi}{\partial x_0} - \frac{\partial}{\partial x_0}\frac{\partial\phi}{\partial x_2}, \frac{\partial}{\partial x_0}\frac{\partial\phi}{\partial x_1} - \frac{\partial}{\partial x_1}\frac{\partial\phi}{\partial x_0}\right) \\
&= (0, 0, 0)
\end{aligned}$$

The last equality is true, because the order of partial differentiation is irrelevant for a function whose second derivatives are continuous.

Exercise 5.11

The proof is

$$
\begin{aligned}
\nabla \cdot (\mathbf{f} \times \mathbf{g}) &= \tfrac{\partial}{\partial x_0}(f_1 g_2 - f_2 g_1) + \tfrac{\partial}{\partial x_1}(f_2 g_0 - f_0 g_2) + \tfrac{\partial}{\partial x_2}(f_0 g_1 - f_1 g_0) \\
&= \tfrac{\partial f_1}{\partial x_0} g_2 + f_1 \tfrac{\partial g_2}{\partial x_0} - \tfrac{\partial f_2}{\partial x_0} g_1 - f_2 \tfrac{\partial g_1}{\partial x_0} + \tfrac{\partial f_2}{\partial x_1} g_0 + f_2 \tfrac{\partial g_0}{\partial x_1} - \tfrac{\partial f_0}{\partial x_1} g_2 - f_0 \tfrac{\partial g_2}{\partial x_1} \\
&\quad + \tfrac{\partial f_0}{\partial x_2} g_1 + f_0 \tfrac{\partial g_1}{\partial x_2} - \tfrac{\partial f_1}{\partial x_2} g_0 - f_1 \tfrac{\partial g_0}{\partial x_2} \\
&= \left(\tfrac{\partial f_2}{\partial x_1} - \tfrac{\partial f_1}{\partial x_2} \right) g_0 + \left(\tfrac{\partial f_0}{\partial x_2} - \tfrac{\partial f_2}{\partial x_0} \right) g_1 + \left(\tfrac{\partial f_1}{\partial x_0} - \tfrac{\partial f_0}{\partial x_1} \right) g_2 \\
&\quad - \left(\tfrac{\partial g_2}{\partial x_1} - \tfrac{\partial g_1}{\partial x_2} \right) f_0 - \left(\tfrac{\partial g_0}{\partial x_2} - \tfrac{\partial g_2}{\partial x_0} \right) f_1 - \left(\tfrac{\partial g_1}{\partial x_0} - \tfrac{\partial g_0}{\partial x_1} \right) f_2 \\
&= (\nabla \times \mathbf{f}) \cdot \mathbf{g} - (\nabla \times \mathbf{g}) \cdot \mathbf{f}
\end{aligned}
$$

Exercise 5.13

The left-hand side of equation (5.21) is

$$\begin{aligned}\nabla(\mathbf{f}\cdot\mathbf{g}) \quad = \quad &(\\ &\tfrac{\partial f_0}{\partial x_0}g_0 + f_0\tfrac{\partial g_0}{\partial x_0} + \tfrac{\partial f_1}{\partial x_0}g_1 + f_1\tfrac{\partial g_1}{\partial x_0} + \tfrac{\partial f_2}{\partial x_0}g_2 + f_2\tfrac{\partial g_2}{\partial x_0}, \\ &\tfrac{\partial f_0}{\partial x_1}g_0 + f_0\tfrac{\partial g_0}{\partial x_1} + \tfrac{\partial f_1}{\partial x_1}g_1 + f_1\tfrac{\partial g_1}{\partial x_1} + \tfrac{\partial f_2}{\partial x_1}g_2 + f_2\tfrac{\partial g_2}{\partial x_1}, \\ &\tfrac{\partial f_0}{\partial x_2}g_0 + f_0\tfrac{\partial g_0}{\partial x_2} + \tfrac{\partial f_1}{\partial x_2}g_1 + f_1\tfrac{\partial g_1}{\partial x_2} + \tfrac{\partial f_2}{\partial x_2}g_2 + f_2\tfrac{\partial g_2}{\partial x_2} \\ &)\end{aligned} \tag{8}$$

The first term of the right-hand side of equation (5.21) is

$$\begin{aligned}\mathbf{f}\times(\nabla\times\mathbf{g}) \quad = \quad &(\\ &f_1\left(\tfrac{\partial g_1}{\partial x_0} - \tfrac{\partial g_0}{\partial x_1}\right) - f_2\left(\tfrac{\partial g_0}{\partial x_2} - \tfrac{\partial g_2}{\partial x_0}\right), \\ &f_2\left(\tfrac{\partial g_2}{\partial x_1} - \tfrac{\partial g_1}{\partial x_2}\right) - f_0\left(\tfrac{\partial g_1}{\partial x_0} - \tfrac{\partial g_0}{\partial x_1}\right), \\ &f_0\left(\tfrac{\partial g_0}{\partial x_2} - \tfrac{\partial g_2}{\partial x_0}\right) - f_1\left(\tfrac{\partial g_2}{\partial x_1} - \tfrac{\partial g_1}{\partial x_2}\right) \\ &)\end{aligned} \tag{9}$$

The second term is

$$\begin{aligned}\mathbf{g}\times(\nabla\times\mathbf{f}) \quad = \quad &(\\ &g_1\left(\tfrac{\partial f_1}{\partial x_0} - \tfrac{\partial f_0}{\partial x_1}\right) - g_2\left(\tfrac{\partial f_0}{\partial x_2} - \tfrac{\partial f_2}{\partial x_0}\right), \\ &g_2\left(\tfrac{\partial f_2}{\partial x_1} - \tfrac{\partial f_1}{\partial x_2}\right) - g_0\left(\tfrac{\partial f_1}{\partial x_0} - \tfrac{\partial f_0}{\partial x_1}\right), \\ &g_0\left(\tfrac{\partial f_0}{\partial x_2} - \tfrac{\partial f_2}{\partial x_0}\right) - g_1\left(\tfrac{\partial f_2}{\partial x_1} - \tfrac{\partial f_1}{\partial x_2}\right) \\ &)\end{aligned} \tag{10}$$

The third term is

$$\begin{aligned}(\mathbf{f}\cdot\nabla)\mathbf{g} \quad = \quad &(\\ &f_0\tfrac{\partial g_0}{\partial x_0} + f_1\tfrac{\partial g_0}{\partial x_1} + f_2\tfrac{\partial g_0}{\partial x_2}, \\ &f_0\tfrac{\partial g_1}{\partial x_0} + f_1\tfrac{\partial g_1}{\partial x_1} + f_2\tfrac{\partial g_1}{\partial x_2}, \\ &f_0\tfrac{\partial g_2}{\partial x_0} + f_1\tfrac{\partial g_2}{\partial x_1} + f_2\tfrac{\partial g_2}{\partial x_2} \\ &)\end{aligned} \tag{11}$$

The fourth term is

$$\begin{aligned}(\mathbf{g}\cdot\nabla)\mathbf{f} \quad = \quad &(\\ &g_0\tfrac{\partial f_0}{\partial x_0} + g_1\tfrac{\partial f_0}{\partial x_1} + g_2\tfrac{\partial f_0}{\partial x_2}, \\ &g_0\tfrac{\partial f_1}{\partial x_0} + g_1\tfrac{\partial f_1}{\partial x_1} + g_2\tfrac{\partial f_1}{\partial x_2}, \\ &g_0\tfrac{\partial f_2}{\partial x_0} + g_1\tfrac{\partial f_2}{\partial x_1} + g_2\tfrac{\partial f_2}{\partial x_2} \\ &)\end{aligned} \tag{12}$$

Adding equations (9) through (12) leads to quite a few cancellations of terms, the end result being equation (8).

Exercise 5.15

In Exercise 5.8, I defined the permutation tensor e_{ijk}. The cross product $\mathbf{g} \times \mathbf{f}$ is represented using tensor index notation as

$$\mathbf{g} \times \mathbf{f} = e_{ijk} g_j f_k$$

where the repeated indices for j and k indicate a summation over those indices. The only free index is i. We may formally replace $\mathbf{g}$ by ∇. For the sake of notation, write ∇ in tensor index notation as ∂_i; that is, the ith component of the tensor is the partial derivative with respect to variable x_i. The curl is written in tensor index notation as

$$\nabla \times \mathbf{f} = e_{ijk} \partial_j f_k$$

Extending this idea to applying the curl twice, we have

$$\nabla \times (\nabla \times \mathbf{f}) = e_{ijk} \partial_j e_{k\ell m} \partial_\ell f_m = e_{ijk} e_{k\ell m} \partial_j \partial_\ell f_m = e_{ijk} e_{k\ell m} f_{m,j\ell}$$

The permutation tensor is a constant with respect to $\mathbf{x}$, so it can be factored outside the partial derivative operator. The indices j, k, ℓ, and m are repeated, so summations occur for those indices. The only free index is i, so the resulting expression represents a vector. As mentioned in the book, the notation $f_{m,j\ell}$ indicates that indices before the comma are components of the tensor. Indices after the comma denote application of partial derivatives. Thus, $f_{m,j\ell}$ is the second-order partial derivative $\partial^2 f_m / \partial x_j \partial x_\ell$.

Maintaining a list of new index names is a bookkeeping irritation, so let us summarize the results so far using only i, j, and k, but with subscripts on j and k to obtain new names. We also use the comma notation to separate components of the tensor from derivatives of the components.

$$\begin{aligned} \nabla \times \mathbf{f} &= e_{ij_1k_1} f_{k_1,j_1} \\ \nabla \times (\nabla \times \mathbf{f}) &= e_{ij_2k_2} e_{k_2j_1k_1} f_{k_1,j_1j_2} \end{aligned}$$

Then

$$\begin{aligned} \nabla \times (\nabla \times (\nabla \times \mathbf{f})) &= e_{ij_3k_3} \partial_{j_3} e_{k_3j_2k_2} e_{k_2j_1k_1} f_{k_1,j_1j_2} \\ &= e_{ij_3k_3} e_{k_3j_2k_2} e_{k_2j_1k_1} \partial_{j_3} f_{k_1,j_1j_2} \\ &= e_{ij_3k_3} e_{k_3j_2k_2} e_{k_2j_1k_1} f_{k_1,j_1j_2j_3} \end{aligned}$$

The tensor index notation and subscripting of the newly introduced indices makes it easy to extend the formula to more applications of the curl:

$$(\nabla\times)^n \mathbf{f} = e_{ij_nk_n} e_{k_nj_{n-1}k_{n-1}} \cdots e_{k_2j_1k_1} f_{k_1,j_1\cdots j_n}$$

If you want to compactify the notation even further, define

$$\mathbf{c} = (\nabla\times)^n \mathbf{f}$$

where $\mathbf{c}$ is represented by the tensor $c_{k_{n+1}}$ with free index k_{n+1}; then

$$c_{k_{n+1}} = \left(\prod_{\ell=1}^{n} e_{k_{\ell+1}j_\ell k_\ell} \right) f_{k_1,j_1\cdots j_n}$$

The large pi represents a product of terms, just as a large sigma represents a sum of terms. Oh well, mathematical notation tends to be concise and elegant while at the same time appearing to be quite obfuscated!

6 Physics Engines

Exercise 6.1. A naive implementation that determines which bins are intersected by a sphere will iterate over all bins and call the sphere-box `TestIntersection` function mentioned on page 302. This requires n intersection tests. If the bins are arranged in a rectangular grid, as illustrated in Figure 6.2, you can speed things up by projection onto the coordinate axes.

If the sphere has center (x_c, y_c, z_c) and radius r, the interval of projection onto the x-axis is $[x_c - r, x_c + r]$, the interval of projection onto the y-axis is $[y_c - r, y_c + r]$, and the interval of projection onto the z-axis is $[z_c - r, z_c + r]$. Knowing that the bins have constant dimensions, you can determine the subset of bins that have the potential to intersect the axis-aligned box $[x_c - r, x_c + r] \times [y_c - r, y_c + r] \times [z_c - r, z_c + r]$. The determination involves computing minimum and maximum x-, y-, and z-indices for the bins, something that requires only a few computations. The subset of bins is typically a lot less than total number, unless the sphere is really large. In the latter case, your bins are most likely too small a partition of space. The cost equation is dependent on the ratio of the number of bins in the subset to the total number of bins. With a good choice of bin size, you should be able to obtain an approximately constant number of intersection tests (order $O(1)$ instead of order $O(n)$).

Exercise 6.3. This is not a question that requires an answer. There are many ways to implement such an algorithm.

Exercise 6.5. The idea is to construct a delta function that corresponds to the set of points $S = \{\mathcal{P}_0, \mathcal{P}_1\}$. This is a matter of choosing an average of two delta functions, each one representing a point in the set. Such a delta function has a property analogous to equation (6.88),

$$\int_{\mathcal{X}} \mathbf{G}(\mathcal{X})\delta(\mathcal{X}, S)d\mathcal{X} = \frac{\mathbf{G}(\mathcal{P}_0) + \mathbf{G}(\mathcal{P}_1)}{2}$$

Equation (6.89) for this delta function is

$$\begin{aligned}
\mathbf{v}_A^+ &= \mathbf{v}_A^- + m_A^{-1} \int_{\mathcal{X}} \int_t f\mathbf{N}\delta(t - t_0)\delta(\mathcal{X}, S)\, dt\, d\mathcal{X} \\
&= \mathbf{v}_A^- + m_A^{-1} \int_{\mathcal{X}} f\mathbf{N}\delta(\mathcal{X}, S)\, d\mathcal{X} \\
&= \mathbf{v}_A^- + m_A^{-1} \tfrac{f\mathbf{N} + f\mathbf{N}}{2} \\
&= \mathbf{v}_A^- + m_A^{-1} f\mathbf{N}
\end{aligned}$$

Equation (6.91) for this delta function is

$$\begin{aligned}
\mathbf{w}_A^+ &= \mathbf{w}_A^- + J_A^{-1} \int_{\mathcal{X}} \int_t \mathbf{r}_A(\mathcal{X}, t) \times f\mathbf{N}\delta(t - t_0)\delta(\mathcal{X}, S)\, dt\, d\mathcal{X} \\
&= \mathbf{w}_A^- + J_A^{-1} \int_{\mathcal{X}} \mathbf{r}_A(\mathcal{X}, t_0) \times f\mathbf{N}\delta(\mathcal{X}, S)\, d\mathcal{X} \\
&= \mathbf{w}_A^- + J_A^{-1} \tfrac{\mathbf{r}_A(\mathcal{P}_0, t_0) \times f\mathbf{N} + \mathbf{r}_A(\mathcal{P}_1, t_0) \times f\mathbf{N}}{2} \\
&= \mathbf{w}_A^- + J_A^{-1} \left(\tfrac{\mathbf{r}_A(\mathcal{P}_0, t_0) + \mathbf{r}_A(\mathcal{P}_1, t_0)}{2}\right) \times f\mathbf{N} \\
&= \mathbf{w}_A^- + J_A^{-1} \mathbf{r}_A(\mathcal{M}, t_0) \times f\mathbf{N}
\end{aligned}$$

where $\mathcal{M} = (\mathcal{P}_0 + \mathcal{P}_1)/2$, the point midway between the two contact points. This is exactly the same result that we had for the line segment of contact. Therefore, the simplifying assumption of considering only a finite set of contact points is not an approximation after all! (Well, in the case of constant density mass, anyway.)

7 Linear Algebra

Exercise 7.1. The cost for row reduction using the first row is still $((2\mu+\alpha)n)(n-1)$. The $(2\mu+\alpha)n$ portion is the time spent recomputing a single row's entries in columns 2 through $n+1$ (recall we are working with the augmented matrix, which is $n \times (n+1)$). There are $n-1$ rows to work with. In the forward elimination of column entries *below* the current row, the cost for the second row operations is $((2\mu+\alpha)(n-1))(n-2)$. The $n-2$ represents the fact that we clear out entries only below row two. However, the full elimination must clear out the column entries above row two, so the cost is modified to $((2\mu+\alpha)(n-1))(n-1)$. In general, the full elimination costs are

$$F_n = (2\mu+\alpha)[n(n-1)+(n-1)(n-1)+\cdots+(1)(n-1)] = (2\mu+\alpha)(n-1)n(n+1)/2$$

The result of full elimination is an augmented matrix for which the first $n \times n$ portion is diagonal. The final phase is to divide each row by the diagonal entry, a total cost of

$$B_n = \delta n$$

The total cost is

$$C'_n = F_n + B_n = (2\mu+\alpha)(n-1)n(n+1)/2 + \delta n$$

The cost for the original algorithm was shown to be

$$C_n = (2\mu+\alpha)(n-1)n(n+1)/3 + (\mu+\alpha)n(n-1)/2 + \delta n$$

The difference is

$$C'_n - C_n = (2\mu+\alpha)(n-1)n(n+1)/6 - (\mu+\alpha)n(n-1)/2 = [n(n-1)/6][(2\mu+\alpha)n - \mu]$$

We already know $n > 1$, so $n(n-1)/6 > 0$. Clearly $C'_n \geq C_n$ when $n \geq \mu/(2\mu+\alpha)$. The right-hand side is a number that is smaller than $1/2$, so for integers we need $n \geq 1$. Consequently, $C'_n \geq C_n$ for all $n \geq 1$. An implementation should use forward elimination, not full elimination, in order to keep the costs to a minimum.

Exercise 7.3. Let $A = [a_{rc}]$ and $B = [b_{rc}]$ be $n \times n$ matrices where r is the row index and c is the column index. The product $AB = [m_{rc}]$ where

$$m_{rc} = \sum_{k=1}^{n} a_{rk} b_{kc}$$

The transposes are defined by $A^{\mathrm{T}} = [a'_{rc}]$, $B = [b'_{rc}]$, and $(AB)^{\mathrm{T}} = [m'_{rc}]$ where $a'_{rc} = a_{cr}$, $b'_{rc} = b_{cr}$, and $m'_{rc} = m_{cr}$. Therefore

$$\begin{aligned} m'_{rc} &= m_{cr} \\ &= \textstyle\sum_{k=1}^{n} a_{ck} b_{kr} \\ &= \textstyle\sum_{k=1}^{n} a'_{kc} b'_{rk} \\ &= \textstyle\sum_{k=1}^{n} b'_{rk} a'_{kc} \end{aligned}$$

The condition $m'_{rc} = \sum_{k=1}^{n} b'_{rk} a'_{kc}$ says that $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$.

Exercise 7.5. Let $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$. Then

$$\begin{aligned}
\mathbf{0} &= \mathbf{u} \times \mathbf{0} \\
&= \mathbf{u} \times (\mathbf{u} + \times \mathbf{v} + \mathbf{w}) \\
&= \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \\
&= \mathbf{0} + \mathbf{u} \times \mathbf{v} - \mathbf{w} \times \mathbf{v}
\end{aligned}$$

Consequently,

$$\mathbf{u} \times \mathbf{v} = \mathbf{w} \times \mathbf{u}$$

A similar construction using a cross product of the equation with $\mathbf{v}$ will lead to

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w}$$

The geometric interpretation is that the three vectors may be viewed as the three edges of a triangle in a plane. The fact that the sum is zero means that as you walk around the triangle boundary, starting and ending at the same vertex, your net distance traveled is zero. The equality of the three cross products says that it does not matter which two edges of the triangle you use to compute a normal vector, and that normal vector always has the same length (which is twice the area of the triangle).

Exercise 7.7.

Item 1. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, any vector may be written as a linear combination of these

$$\mathbf{p} = x\mathbf{u} + y\mathbf{v} + z\mathbf{w}$$

Then

$$\begin{aligned} \mathbf{p} \cdot \mathbf{v} \times \mathbf{w} &= x\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} \\ \mathbf{p} \cdot \mathbf{w} \times \mathbf{u} &= y\mathbf{v} \cdot \mathbf{w} \times \mathbf{u} \\ \mathbf{p} \cdot \mathbf{u} \times \mathbf{v} &= z\mathbf{w} \cdot \mathbf{u} \times \mathbf{v} \end{aligned}$$

The first equation has solution

$$x = \frac{\mathbf{p} \cdot \mathbf{v} \times \mathbf{w}}{\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}}$$

The second equation has solution

$$y = \frac{\mathbf{p} \cdot \mathbf{w} \times \mathbf{u}}{\mathbf{v} \cdot \mathbf{w} \times \mathbf{u}} = \frac{\mathbf{u} \cdot \mathbf{p} \times \mathbf{w}}{\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}}$$

The third equation has solution

$$z = \frac{\mathbf{p} \cdot \mathbf{u} \times \mathbf{v}}{\mathbf{w} \cdot \mathbf{u} \times \mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v} \times \mathbf{p}}{\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}}$$

Item 2. Write $\mathbf{p}$ as a linear combination of the three pairs of cross products,

$$\mathbf{p} = x\mathbf{u} \times \mathbf{v} + y\mathbf{v} \times \mathbf{w} + z\mathbf{w} \times \mathbf{u}$$

Then

$$\begin{aligned} \mathbf{w} \cdot \mathbf{p} &= x\mathbf{w} \cdot \mathbf{u} \times \mathbf{v} \\ \mathbf{u} \cdot \mathbf{p} &= y\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} \\ \mathbf{v} \cdot \mathbf{p} &= z\mathbf{v} \cdot \mathbf{w} \cdot \mathbf{u} \end{aligned}$$

The first equation has solution

$$x = \frac{\mathbf{w} \cdot \mathbf{p}}{\mathbf{w} \cdot \mathbf{u} \times \mathbf{v}} = \frac{\mathbf{w} \cdot \mathbf{p}}{\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}}$$

The second equation has solution

$$y = \frac{\mathbf{u} \cdot \mathbf{p}}{\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}}$$

The third equation has solution

$$z = \frac{\mathbf{v} \cdot \mathbf{p}}{\mathbf{v} \cdot \mathbf{w} \times \mathbf{u}} = \frac{\mathbf{v} \cdot \mathbf{p}}{\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}}$$

Exercise 7.9. The formula for the determinant of a 4×4 matrix is

$$\det(A) = \sum_{\sigma} = \varepsilon_{\sigma} a_{\sigma(1)1} a_{\sigma(2)2} a_{\sigma(3)3} a_{\sigma(4)4}$$

where $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ is a permutation of $(1, 2, 3, 4)$. There are $4! = 24$ such permutations, so the determinant has 24 terms, each term having four a_{ij} terms and a plus or a minus sign. The table below shows the 24 terms and signs.

permutation	term	permutation	term
$(1,2,3,4)$	$+a_{11}a_{22}a_{33}a_{44}$	$(1,4,2,3)$	$+a_{11}a_{42}a_{23}a_{34}$
$(3,1,2,4)$	$+a_{31}a_{12}a_{23}a_{44}$	$(3,4,1,2)$	$+a_{31}a_{42}a_{13}a_{24}$
$(2,3,1,4)$	$+a_{21}a_{32}a_{13}a_{44}$	$(2,4,3,1)$	$+a_{21}a_{42}a_{33}a_{14}$
$(1,3,2,4)$	$-a_{11}a_{32}a_{23}a_{44}$	$(1,4,3,2)$	$-a_{11}a_{42}a_{33}a_{24}$
$(3,2,1,4)$	$-a_{31}a_{22}a_{13}a_{44}$	$(3,4,2,1)$	$-a_{31}a_{42}a_{23}a_{14}$
$(2,1,3,4)$	$-a_{21}a_{12}a_{33}a_{44}$	$(2,4,1,3)$	$-a_{21}a_{42}a_{13}a_{34}$
$(1,2,4,3)$	$-a_{11}a_{22}a_{43}a_{34}$	$(4,1,2,3)$	$-a_{41}a_{12}a_{23}a_{34}$
$(3,1,4,2)$	$-a_{31}a_{12}a_{43}a_{24}$	$(4,3,1,2)$	$-a_{41}a_{32}a_{13}a_{24}$
$(2,3,4,1)$	$-a_{21}a_{32}a_{43}a_{14}$	$(4,2,3,1)$	$-a_{41}a_{22}a_{33}a_{14}$
$(1,3,4,2)$	$+a_{11}a_{32}a_{43}a_{24}$	$(4,1,3,2)$	$+a_{41}a_{12}a_{33}a_{24}$
$(3,2,4,1)$	$+a_{31}a_{22}a_{43}a_{14}$	$(4,3,2,1)$	$+a_{41}a_{32}a_{23}a_{14}$
$(2,1,4,3)$	$+a_{21}a_{12}a_{43}a_{34}$	$(4,2,1,3)$	$+a_{41}a_{22}a_{13}a_{34}$

Exercise 7.11. Let U be the upper triangular matrix with diagonal entries u_{ii} for $1 \leq i \leq n$. To compute the determinant, use a cofactor expansion by the *last row*. Since the first $n-1$ entries of the row are zero, there is no need to compute the determinant of the corresponding $(n-1) \times (n-1)$ submatrices. The only possibly nonzero contribution to the determinant comes from the entry u_{nn} times the determinant of the submatrix consisting of the first $n-1$ rows and first $n-1$ columns of U. This submatrix, call it U' is also upper triangular. Thus,

$$\det(U) = u_{nn} \det(U')$$

Notice that regardless of n, the sign term of u_{nn} in the cofactor expansion is always a $+1$. The same approach works for computing the determinant of U', so

$$\det(U) = u_{n,n} u_{n-1,n-1} \cdots u_{1,1}$$

which is the product of the diagonal entries.

8 Affine Algebra

Exercise 8.1. In the construction of Example 8.1, the only thing that changes when the two origins are the same is that $(c_1, c_2) = (0, 0)$. The final relationship between the coordinates is

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = R^{\mathrm{T}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where R is the counterclockwise rotation matrix by an angle $\pi/4$.

Exercise 8.3. The affine subspaces are not parallel. The vector space associated with A_1 is the set of vectors on the x-axis, V_1. The vector space associated with A_2 is the set of vectors on the y-axis, V_2. Neither $V_1 \subseteq V_2$ nor $V_2 \subseteq V_1$.

Exercise 8.5. The construction is similar to that in Exercise 8.4. Let the known points be $\mathbf{P}_i = (x_i, y_i, z_i)$ for $0 \leq i \leq 3$. Let $\mathbf{P} = (x, y, z)$. This may be written in barycentric coordinates as

$$\mathbf{P} = c_0\mathbf{P}_0 + c_1\mathbf{P}_1 + c_2\mathbf{P}_2 + c_3\mathbf{P}_3$$

where $c_0 + c_1 + c_2 + c_3 = 1$. Since f is affine,

$$f(\mathbf{P}) = c_0 f(\mathbf{P}_0) + c_1 f(\mathbf{P}_1) + c_2 f(\mathbf{P}_2) + c_3 f(\mathbf{P}_3) = c_0 f_0 + c_1 f_1 + c_2 f_2 + c_3 f_3$$

The value $f(\mathbf{P})$ is known once we compute c_0, c_1, c_2, and c_3.

Define $\mathbf{V}_i = \mathbf{P}_i - \mathbf{P}_0$ for $1 \leq i \leq 3$ and $\mathbf{V} = \mathbf{P} - \mathbf{P}_0$. Subtracting $\mathbf{P}_0$ from the first displayed equation and using $c_0 - 1 = -c_1 - c_2 - c_3$,

$$\mathbf{V} = c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + c_3\mathbf{V}_3$$

Convert to a linear system,

$$\begin{bmatrix} \mathbf{V}_1 \cdot \mathbf{V}_1 & \mathbf{V}_1 \cdot \mathbf{V}_2 & \mathbf{V}_1 \cdot \mathbf{V}_3 \\ \mathbf{V}_2 \cdot \mathbf{V}_1 & \mathbf{V}_2 \cdot \mathbf{V}_2 & \mathbf{V}_2 \cdot \mathbf{V}_3 \\ \mathbf{V}_3 \cdot \mathbf{V}_1 & \mathbf{V}_3 \cdot \mathbf{V}_2 & \mathbf{V}_3 \cdot \mathbf{V}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 \cdot \mathbf{V} \\ \mathbf{V}_2 \cdot \mathbf{V} \\ \mathbf{V}_3 \cdot \mathbf{V} \end{bmatrix}$$

The 3×3 coefficient matrix is invertible since the $\mathbf{V}_i$ are linearly independent (edges of a tetrahedron with common end point $\mathbf{P}_0$). Solve the matrix with a standard Gaussian elimination.

9 Calculus

Exercise 9.1. Let the triangle vertices be $\mathbf{V}_i$ for $0 \le i \le 2$. Any point on the triangle is of the form

$$\mathbf{V}(s,t) = \mathbf{V}_0 + s(\mathbf{V}_1 - \mathbf{V}_0) + t(\mathbf{V}_2 - \mathbf{V}_0) = \mathbf{V}_0 + s\mathbf{E}_1 + t\mathbf{E}_2$$

where $s \ge 0$, $t \ge 0$, and $s + t \le 1$. This is just an application of barycentric coordinates, the coordinates being $1 - s - t$, s, and t relative to the three vertices. The distance from a point $\mathbf{P}$ to the triangle is the minimum distance between $\mathbf{P}$ and all points $\mathbf{V}(s,t)$ on the triangle. The values of (s,t) are selected so the squared distance $f(s,t)$ is minimimized. The squared distance is listed below with $\mathbf{D} = \mathbf{V}_0 - \mathbf{P}$,

$$\begin{aligned}
f(s,t) &= |\mathbf{V}(s,t) - \mathbf{P}|^2 \\
&= |s\mathbf{E}_1 + t\mathbf{E}_2 + \mathbf{D}|^2 \\
&= |\mathbf{E}_1|^2 s^2 + 2(\mathbf{E}_1 \cdot \mathbf{E}_2)st + |\mathbf{E}_2|^2 t^2 + 2(\mathbf{E}_1 \cdot \mathbf{D})s + 2(\mathbf{E}_2 \cdot \mathbf{D})t + |\mathbf{D}|^2 \\
&= a_{00}s^2 + 2a_{01}st + a_{11}t^2 + 2b_0 s + 2b_1 t + c
\end{aligned}$$

which is exactly the function analyzed in Example 9.8.

10 Quaternions

No exercises.

11 Differential Equations

No exercises.

12 Ordinary Difference Equations

No exercises.

13 Numerical Methods

No exercises.

14 Linear Complementarity and Mathematical Programming

No exercises.